

# Extended Permutation Coding with Spectral Shaping Capability

K. Ogunyanda, T. G. Swart, and O. O. Ogunyanda

Center for Telecommunications, Department of Electrical and Electronic Engineering Science, University of Johannesburg, Johannesburg, South Africa

Email: ogunyanda@gmail.com; tgswart@uj.ac.za; opeyemiogunyanda@gmail.com

**Abstract**—Considerable effort has been focused on efficient error correction schemes in digital communications where the medium of transmission is unfriendly for information transmission. Also, some of such channels have certain frequency characteristics that should be considered during system designs. Permutation coding (PC) has been reported as being capable of spectral shaping, by manipulating channel symbols in such a way that the frequency energy of the transmitted signal is concentrated towards certain areas of the frequency spectrum. We, therefore, report an extension operation that enables the generation of a new PC from a source PC that is capable of spectrum shaping, while still maintaining its spectral shaping capability. A mathematical expression that can predict the low energy sections of the codebook's spectrum is also presented, together with their lower and upper bounds. An added advantage of the proposed scheme is that better error correction is attainable due to the way the extension is introduced.

**Index Terms**—Channel coding, code extension, error correction, permutation codes, power line communications, spectral shaping

## I. INTRODUCTION

In a bid to overcome communication challenges such as zero frequency components, baseline wanders and the need to meet specific spectrum behavior in communication designs, it is pertinent to involve the use of coded modulation schemes that can assist in directing a signal's spectrum energy towards certain sections of the spectrum. This is otherwise known as spectral shaping. In the literature, a number of spectrum shaping schemes have been reported [1]–[6]. Non-binary Permutation Coding (PC) sequences were used in [1] to manipulate PAM channel symbols to achieve spectral shaping. The spectrum of the sequences exhibit spectral nulls at rational sub-multiples of the symbol frequency. Also, the authors in [2] reported a special form of PC called the injection coding in concatenation with PAM symbols for achieving spectral shaping. The injection coding was derived from a conventional PC by shortening its codeword length.

In channel coding, the Hamming distance  $d$ , which is the pairwise comparison of the positions of individual code symbols in two codewords of the same codebook, is

used to define the error-correcting capability of a codebook [7]. One major challenge in spectral shaping PCs is, however, the low minimum Hamming distance  $d_{\min}$ , which is the minimum value of  $d$  obtainable from a codebook  $C$ . For instance, all the spectral shaping PCs reported in [5] have a  $d_{\min}$  of 2. Despite the fact that such codebooks are capable of shaping the spectrums of the transmitted signals, so as to achieve certain design requirements, the robustness of such a system against channel errors is negatively affected. With a view to fostering the robustness of spectral shaping PCs, we therefore report a new approach, which can improve the  $d_{\min}$  of such codebooks.

There are a number of ways to improve the Hamming distance profile of codebooks. One way to achieve this is the use of the upper bound of the codebook's cardinality  $|C|$  to construct the codebook while the  $d_{\min}$  is predetermined. Such a bound is defined using the relationship [8]:

$$|C| \leq \frac{M!}{(d_{\min} - 1)!}, \quad (1)$$

where  $M$  is the codeword length. However, other constraints, other than the  $d_{\min}$ , are considered in constructing spectral shaping codebooks so as to meet the desired spectrum pattern. Code extension is another approach that can be employed in improving codebooks' Hamming distance profile [9]–[13]. For example, a partition and extension approach was reported in [10] to produce new permutation arrays with better Hamming distance. Code extension has also been reported for applications such as audio coding as reported in [14] and [15].

The approach reported in this paper, therefore, involves extending the  $M$  of a source PC that is capable of spectral shaping. Due to the fact that the codeword length of the target codebook (denoted by  $M'$ ) is greater than the  $M$  of the source PC, we are able to attain higher  $d_{\min}$ , if the extension is properly introduced, and, at the same time, retain the spectral shaping capability of the source PC. An expression, which can be used to determine the possible spectrum variation resulting from the extension symbols in the codebook, is also derived. A similar expression has been reported in [2], but in the context of injection/puncturing. As such, this work can be regarded as an additional work on [2].

---

Manuscript received November 21, 2022; revised May 17, 2022.  
Corresponding author email: ogunyanda@gmail.com.  
doi:10.12720/jcm.17.6.434-442

Section II of this paper provides a brief description of permutation coding and error-correcting capability in relation to  $d_{\min}$ . In Section III, we describe how spectral shaping can be achieved using PAM modulated PC symbols. The proposed extended permutation coding scheme is described in Section IV, and using propositions, analyses on the spectral attributes of the proposed scheme are presented therein. In Section V, we present some results obtained from comparing the spectrums and performances of the proposed scheme and those of the conventional spectral shaping PCs. We conclude the paper in Section VI.

## II. PERMUTATION CODING

Channel coding, otherwise known as forward error-correction coding (FEC), is the process of introducing redundant/overhead information in the transmitted signal, so as to mitigate against the channel impairments. This is achieved by transforming the useful user data from one form to another. In permutation coding (PC), binary user data is transformed into non-binary sequence of codewords, each containing  $M$  non-repetitive symbols [16]–[19]. The value of  $M$  is also regarded as the codeword length.

A typical mapping of binary data onto PC symbols is

$$\left\{ \begin{array}{cccc} \text{binary data} & & & \\ 00 & 10 & 01 & 11 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2\ 3\ 1\ 0 & 1\ 3\ 2\ 0 & 2\ 0\ 1\ 3 & 1\ 0\ 2\ 3 \\ \text{PC symbols} & & & \end{array} \right\} \quad (2)$$

The cardinality  $|\mathcal{C}|$  of a codebook is the total number of codewords therein, and it is upper bounded by (1). Given a PC codebook with codeword length  $M$ , symbol size  $q$  and cardinality  $|\mathcal{C}|$ , the code rate  $R$  is given by

$$R = \frac{\log_2 |\mathcal{C}|}{M \log_2 q}. \quad (3)$$

The numerator of (3) is the number of bits mapped to  $M$  PC symbols, while the denominator is the bit equivalent of the PC symbols. Using (2) as an example, we see that  $M = q = 4$  and  $|\mathcal{C}| = 4$ . The code rate is, therefore,

$$R = \frac{\log_2 4}{4 \log_2 4} = \frac{2}{8} = 0.25. \quad (4)$$

This implies that 2 bits are mapped to 4 PC symbols whose bit equivalent is 8 bits. The redundant bits introduced here is given by

$$M \log_2 q - \log_2 |\mathcal{C}|, \quad (5)$$

which, in this example is  $8 - 2 = 6$ .

In terms of the  $d_{\min}$ , a codebook is capable of detecting  $t$  errors according to the expression [7]

$$t \leq d_{\min} - 1, \quad (6)$$

and it is capable of correcting  $t/2$  errors. Hence, a system with improved error-correcting capability is achievable if its  $d_{\min}$  is increased, but at the cost of decreasing the code rate [20].

### A. PC Distance Profile

Let us denote  $d_{i,j}$  as the Hamming distance between the  $i$ -th and  $j$ -th codewords in a permutation codebook. The distance profile of each codeword (i.e., the various values of distances  $d_{i,j}$  contributed by the codeword) can be analysed by representing these  $d_{i,j}$  in a matrix denoted by  $\mathbf{E}$ . Let us demonstrate this using the PC  $\{0\ 1\ 2\ 3, 1\ 2\ 3\ 0, 0\ 1\ 3\ 2, 3\ 0\ 1\ 2\}$ . Its matrix  $\mathbf{E}$  is given by

$$\mathbf{E} = \begin{matrix} & \begin{matrix} 0123 & 1230 & 0132 & 3012 \end{matrix} \\ \begin{matrix} 0123 \\ 1230 \\ 0132 \\ 3012 \end{matrix} & \begin{pmatrix} 0 & 4 & 2 & 4 \\ 4 & 0 & 3 & 4 \\ 2 & 3 & 0 & 3 \\ 4 & 4 & 3 & 0 \end{pmatrix} \end{matrix}. \quad (7)$$

From the matrix in (7), codeword  $\{0\ 1\ 3\ 2\}$ , for instance, contributes distance 2 once, and distance 3 twice. By considering all the codewords, we can infer that cumulatively, distance 2 appears 2 times, distance 3 appears 4 times, while distance 4 appears 6 times.

### B. PC Distance Profile Optimization

Using the distance profile depicted by the matrix  $\mathbf{E}$  of a codebook, we are able to optimize codebooks by replacing any codeword that contributes unwanted distance in the codebook. This is best illustrated using the sample codebook from (7).

By observation, we see that the codeword  $\{0\ 1\ 3\ 2\}$  contributes non-optimal distances 2 and 3, while all other codewords contribute only distance 4. In order to optimize the distance profile, the symbols of the codeword  $\{0\ 1\ 3\ 2\}$  need to be re-permuted so as to get rid of the unwanted distances. In this regard, the best symbol permutation of the codeword is  $\{2\ 3\ 0\ 1\}$ , which contributes only distance 4. The resulting matrix  $\mathbf{E}$  is given by

$$\mathbf{E} = \begin{matrix} & \begin{matrix} 0123 & 1230 & 2301 & 3012 \end{matrix} \\ \begin{matrix} 0123 \\ 1230 \\ 2301 \\ 3012 \end{matrix} & \begin{pmatrix} 0 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 \\ 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & 0 \end{pmatrix} \end{matrix}. \quad (8)$$

## III. SPECTRAL SHAPING USING PERMUTATION CODING

In data transmission, spectral shaping is said to be achieved if data-containing alphabets can be permuted in such a way that a considerable amount of the signal energy is directed towards a predetermined range of the frequency spectrum. If the spectrum is zero at specific equal intervals of the frequency, spectral nulls are said to be achieved.

Given a permutation codebook  $\mathcal{C}$  comprising of  $|\mathcal{C}|$  codewords/sequences  $C_\tau = \{c_0, c_1, \dots, c_{M-1}\}$ , with  $M$  being their word length, each element  $c_i$  in  $C_\tau$  can be mapped to a corresponding channel symbol  $x_i$ . With this, the amplitude-phase frequency spectrum  $s_\tau$  of the discrete signal is given by [21]

$$s_\tau = \sum_{i=0}^{M-1} x_i e^{-j2\pi f T(i+1)}, \quad (9)$$

where  $T$  is the duration of the channel symbol  $x_i$ . Since  $\mathcal{C}$  is composed of  $|\mathcal{C}|$  sequences, its average power spectral density (PSD) can be expressed as

$$S = \frac{1}{|\mathcal{C}|M} \sum_{\tau=1}^{|\mathcal{C}|} |s_\tau|^2. \quad (10)$$

To make  $S$  become 0 (i.e., spectral null) at certain chosen frequencies,  $M$  is made to be a multiple of an integer  $G$  during the code design [22]. This implies that

$$M = gG, \quad (11)$$

with  $g$  being an integer as well. With this, we can say that each  $C_\tau$  consists of symbols  $c_i$  grouped into  $G$  groups, each consisting of  $g$  symbols. Hence, using a normalized frequency spectrum, the codebook exhibits spectral nulls at  $k$  multiples of  $1/G$ , where  $k = 0, 1, \dots, G$ , if the expression [21], [22]

$$A_0 = A_1 = \dots = A_{G-1} \quad (12)$$

is satisfied, where

$$A_i = \sum_{n=0}^{g-1} x_{i+nG}, \quad i = 0, 1, \dots, G-1. \quad (13)$$

To achieve spectral shaping, any modulation scheme, whose constellations are symmetrical, can be employed. As done in [2], [3] and [5], PAM symbols were combined with permutation schemes to achieve spectral shaping. Also, QAM constellations are symmetrical since its symbols can be built from two orthogonal PAM components. Thus, this makes it suitable for a similar application. However, for the purpose of simplicity, we shall stick to a PAM system in this work.

As presented in [2], provided that even symbol size  $q$  is involved, PC symbols can be mapped to PAM channel symbols using

$$\left\{ \begin{array}{cccccccc} 0 & 1 & \dots & \frac{M}{2}-1 & \frac{M}{2} & \dots & M-2 & M-1 \\ \downarrow & \downarrow & \dots & \downarrow & \downarrow & \dots & \downarrow & \downarrow \\ -\frac{M-1}{2} & -(\frac{M-1}{2}-1) & \dots & -\frac{1}{2} & \frac{1}{2} & \dots & \frac{M-1}{2}-1 & \frac{M-1}{2} \end{array} \right\}, \quad (14)$$

where, in PC,  $M = q$ .

*Example 1:* Consider a permutation codebook as in (2) with its corresponding PAM symbol mappings given by

$$\left\{ \begin{array}{cccc} c_0 & c_1 & c_2 & c_3 \\ C_1 : & 2 & 3 & 1 & 0 \\ C_2 : & 1 & 3 & 2 & 0 \\ C_3 : & 2 & 0 & 1 & 3 \\ C_4 : & 1 & 0 & 2 & 3 \end{array} \right\} \Rightarrow \left\{ \begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \\ X_1 : & +0.5 & +1.5 & -0.5 & -1.5 \\ X_2 : & -0.5 & +1.5 & +0.5 & -1.5 \\ X_3 : & +0.5 & -1.5 & -0.5 & +1.5 \\ X_4 : & -0.5 & -1.5 & +0.5 & +1.5 \end{array} \right\}. \quad (15)$$

By applying (13) to codeword  $C_1$  whose channel sequence is  $X_1$ , we have that

$$\begin{aligned} A_0 &= x_0 + x_2 = -1.5 + 1.5 = 0 \text{ and} \\ A_1 &= x_1 + x_3 = -0.5 + 0.5 = 0, \end{aligned} \quad (16)$$

if  $g = 2$  and  $G = 2$ . The same solution applies to the other codewords  $C_2$  to  $C_4$ . Since  $A_0 = A_1$ , this implies that the codebook in (15) will exhibit spectral nulls at  $k$  sub-multiples of  $1/G$ , where  $k = 0, 1, 2$ . This is evident in the spectrum presented in Fig. 1 using a normalized frequency range.

Let us denote the column indices of the codebook as  $i$ , where  $i = 0, 1, \dots, M-1$ . A close look at (15) shows that all channel symbols in column  $i = 0$  are the exact complement of the corresponding symbols in column  $i = 2$ . Also, all symbols in column  $i = 1$  are the complements of those in column  $i = 3$ . This implies that the first half (i.e.,  $M/2$ ) of the entire columns in the codebook is the exact complement of the remaining half. This may not be applicable for cases where  $q$  is not even.

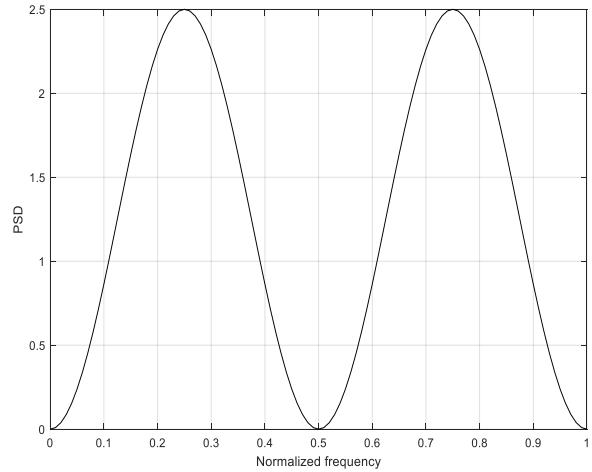


Fig. 1. The spectrum of an  $M = 4$  PC.

#### IV. EXTENDED PERMUTATION CODING AND SPECTRAL SHAPING

Having discussed the significance of  $d_{\min}$  in Section II, the essence of code extension is to increase the value of this variable, so as to improve the codebook's error-correction capability. Here, we make use of a simple algorithm which involves generating and appending additional code symbol(s) to every codeword of a source permutation codebook. In other words, additional column(s) is/are appended to a source codebook to derive a new target codebook. This is as illustrated in Fig. 2 using the same codebook in (2) as the source PC.

The symbols in the appended column are selected such that the summation of all elements in the column is the

same as the summation for each of the other columns. Another criteria considered is to ensure that the resulting  $d'_{\min}$  of the target codebook is  $d'_{\min} \geq d_{\min} + \delta$ , where  $\delta$  is the number of column(s) appended, and any variable with the superscript (') stands for extended PC. This is achieved by observing the matrix  $\mathbf{E}$  of the codebook and then iteratively substituting the appended symbol per each codeword, such that its contribution in the codebook does not make the  $d'_{\min} < d_{\min} + \delta$ , as described in the distance optimization approach presented in Section II.

A. Relationship between PC and Extended PC

By considering the scenario in Fig. 2, we can see that the source PC is of a codeword length  $M = 4$  with a minimum Hamming distance  $d_{\min} = 2$ , while that of the target codebook is  $M' = 5$  with  $d'_{\min} = 3$ . It should however be noted that the resulting codebook's rate is reduced if  $M$  is increased, while  $|\mathcal{C}|$  is fixed. By using (3), we can compute the rate loss  $R_{\text{loss}}$  in % as

$$R_{\text{loss}} = \frac{R - R'}{R} \times 100\%. \quad (17)$$

Rate losses for different scenarios of  $M' = M + \delta$ , with  $|\mathcal{C}'| = |\mathcal{C}|$ , are depicted in Fig. 3. As  $M$  increases, the rate loss decreases. Thus, we can generalize the relationship between a source PC and an extended codebook in terms of codeword length  $M$ , minimum Hamming distance  $d_{\min}$ , cardinality  $|\mathcal{C}|$ , code rate  $R$  and symbol size  $q$  using the expression

$$f(H'_{\min}, H_{\min}, M', M, R', R, |\mathcal{C}'|, |\mathcal{C}|) \equiv \begin{cases} d'_{\min} > d_{\min}, \\ |\mathcal{C}'| = |\mathcal{C}|, \\ R' < R \text{ and} \\ M' > M \text{ if } q' = q, \end{cases} \quad (18)$$

B. Spectrum Attributes of Extended Spectral Shaping PC

Since an extended PC can be derived from a spectral shaping PC, the challenge is to determine if the spectral shaping capability can be retained or not. We shall continue to use the example codebooks presented in Fig. 2 for the purpose of illustration. After mapping the target PC to its corresponding channel symbols, the resulting spectrum is shown as the second curve (dashed line) in Fig. 4. According to the figure, the amplitude of the spectrum of the source PC is approximately zero at certain frequency intervals defined by  $k/G$ , where  $k = 0, 1, 2$  and  $G = 2$ . In other words, the spectrum curve is touching the frequency axis (i.e., zero amplitude) at  $k$  sub-multiples of  $1/G$ . At any other region in the spectrum, the amplitudes are non-zero. In the case of the extended PC, the spectrum exhibit low-energy contents (i.e., notches) at the same sub-multiples of  $1/G$  with an elevated offset above the frequency axis. This offset is termed the frequency variation  $\vartheta$ , as defined in [2] and [3].

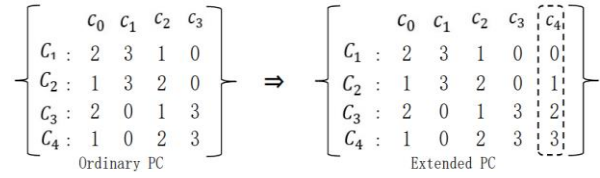


Fig. 2. Extended PC.

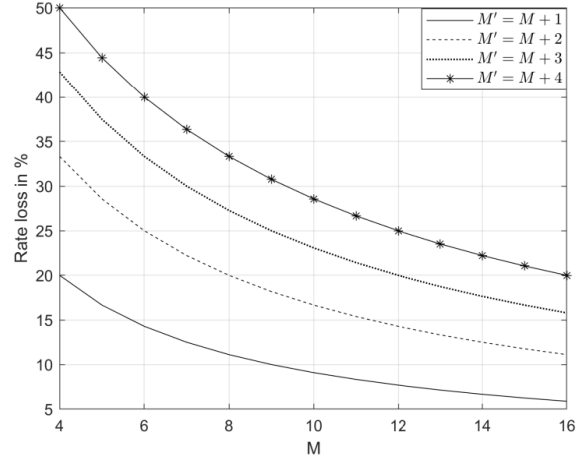


Fig. 3. Rate losses for extended PCs at various  $M'$  values.

Since the spectrums of the two codebooks have low energy contents at frequency sub-multiples of  $1/G$ , they are both said to have spectral shaping capabilities. Additionally, the source PC is also considered a spectral null codebook, since its spectrum energy is approximately zero at  $k$  multiples of  $1/G$ , but the extended PC is not.

**Proposition 1:** Given an  $M$  spectral null PC with notches at frequencies  $f = k/G$  sub-multiples, where  $k = 0, 1, \dots, G$ , it is possible to generate an  $M' = M + \delta$  extended PC that has notches at the same sub-multiples. The notches for such extended PC exhibit a spectrum variation  $\vartheta$  above zero energy frequency, which is given by

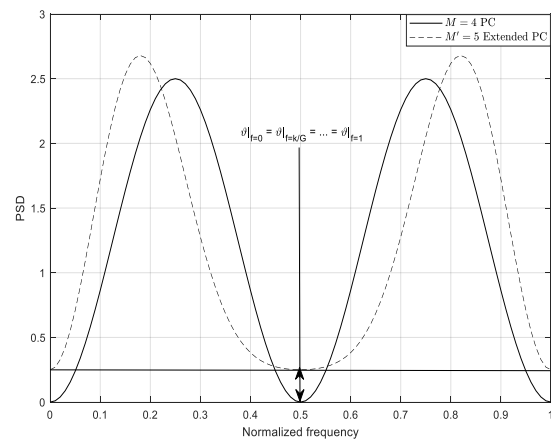


Fig. 4. The spectrums of  $M = 4$  PC and  $M' = 5$  extended PC.

$$\vartheta|_{f=k/G} = \frac{1}{|\mathcal{C}|(M + \delta)} \sum_{\tau=1}^{|\mathcal{C}|} |s'_{\tau}|^2, \quad (19)$$

where

$$s'_\tau = \sum_{i=h}^{(h+\delta)-1} x_i e^{-j2\pi f T(i+1)}, \quad (20)$$

$\delta$  is the number of columns appended to the source PC,  $h$  is the index of the appended column, and  $k = 0, 1, \dots, G$ .

*Proof:* By comparing (19) with (9), we observe that (19) represents the spectrum of  $\delta$  number of columns appended to the source PC. As identified in Section III, each column of the source PC has its exact complement in the entire codebook after being mapped to channel symbols, with  $q$  being even. This implies that, at  $f = k/G$ , the spectral null source PC will have  $s_\tau = 0$ . However, if additional  $\delta$  column(s) is/are appended, this implies that the solution to (9) is based on only the appended column(s), since all  $M$  columns of the source PC reduces to zero at  $f = k/G$ . In other words, the spectrum of the resulting target extended PC, if evaluated at  $f = k/G$ , is the same as the spectrum of the column appended, which is the variation  $\vartheta$  above zero in the spectrum. This thus proves (19).

**Proposition 2:** At  $f = k/G$  frequency sub-multiples, the spectrum variation  $\vartheta$  of a target extended PC has a minimum and maximum possible value bounded by

$$\frac{1}{4(M+1)} \leq \vartheta|_{f=k/G} \leq \frac{M+1}{4}, \quad (21)$$

if only  $\delta = 1$  column is appended, where  $k = 0, 1, \dots, G$ .

*Proof:* As done in [2], we can employ the mapping expression in (14) to prove this. For an even symbol size  $q$ , where  $q = M$ , the minimum absolute channel symbol, according to (14), will be  $|-1/2|$  or  $|+1/2|$ , which both have the same value. Also, the maximum absolute channel symbol can either be  $|(M-1)/2|$  or  $|+(M-1)/2|$ , which are also of the same value.

Now, considering the case where  $\delta = 1$  column is added to a source PC to generate the target extended PC, the lowest spectrum notch that is obtainable, in the best case scenario, will be when all symbols in the appended column are a mixture of only  $-1/2$  and  $+1/2$ . Based on this, (19) can be evaluated at any  $f = k/G$  frequency as

$$\frac{L(1/2)^2}{L(M+1)} = \frac{(1/2)^2}{(M+1)} = \frac{1}{4(M+1)}, \quad (22)$$

since  $\delta = 1$ . We can then infer that any channel symbol combination in the appended column can not yield any spectrum notch value (i.e.,  $\vartheta$  that is less than (22)).

Also, in the worst case scenario, if the column appended to the source PC contains a mixture of only  $-(M-1)/2$  and  $+(M-1)/2$ , we can evaluate (19) at any  $f = k/G$  frequencies as

$$\frac{L((M+1)/2)^2}{L(M+1)} = \frac{((M+1)/2)^2}{(M+1)} = \frac{(M+1)^2}{4(M+1)} = \frac{(M+1)}{4}, \quad (23)$$

since  $\delta = 1$ . Thus, we can infer that any channel symbol combination in the appended column can not yield any notch value that is greater than (23) at  $f = k/G$  sub-frequencies. Hence, both (22) and (23) thus prove (21) in the proposition.

In the event that  $\delta > 1$  columns are appended to the source PC, Proposition 2 may not be applicable. For example, assuming  $\delta = 2$ , it is possible that one of the appended columns is an exact complement of the other. This means that the solution of (19) reduces to zero, thereby causing the spectrum notches to remain on the zero level as in the case of the source PC. Further work on cases where  $\delta > 2$  is left for future work.

### V. ANALYSIS AND PERFORMANCE RESULTS

Here, we shall present some spectral null PCs, out of which corresponding target extended PCs are derived with emphasis on cases where  $\delta = 1$ . Their rates  $R'$ , error-detecting capabilities  $t$  and spectrum curves shall be presented and discussed. Henceforth we shall denote a spectral null PC by a notation  $\mathcal{C}(M,G)$  and the corresponding extended PC by  $\mathcal{C}'(M',G,h)$ , where  $h$  is the column position of the appended symbols.

Table I shows a  $\mathcal{C}(4,2)$  spectral null PC together with its extended versions  $\mathcal{C}'(5,2,4)$  and  $\mathcal{C}'(5,2,0)$ . As shown in the table,  $\mathcal{C}(4,2)$  is able to detect  $t = 1$  error, according to (6), since  $d_{\min} = 2$ . Also, since  $d'_{\min} = 3$ ,  $\mathcal{C}'(5,2,4)$  is capable of detecting  $t' = 2$  errors. Using (3), the rates for these two codebooks are computed as

TABLE I:  $M = 4, G = 2$  SPECTRAL NULL PC AND CORRESPONDING TARGET EXTENDED PC WITH SPECTRAL SHAPING

$\mathcal{C}(4,2)$	$\mathcal{C}'(5,2,4)$	$\mathcal{C}'(5,2,0)$
$d_{\min} = 2$	$d'_{\min} = 3$	$d'_{\min} = 3$
$q = 4$	$q' = 4$	$q' = 4$
0 1 3 2	0 1 3 2 0	0 0 1 3 2
1 0 2 3	1 0 2 3 1	1 1 0 2 3
0 2 3 1	0 2 3 1 2	2 0 2 3 1
2 0 1 3	2 0 1 3 2	2 2 0 1 3
3 1 0 2	3 1 0 2 1	1 3 1 0 2
1 3 2 0	1 3 2 0 0	0 1 3 2 0
3 2 0 1	3 2 0 1 3	3 3 2 0 1
2 3 1 0	2 3 1 0 3	3 2 3 1 0

$$R = \frac{\log_2 |\mathcal{C}|}{M \log_2 q} = \frac{\log_2 8}{4 \log_2 4} = 0.375 \text{ and} \quad (24)$$

$$R' = \frac{\log_2 |\mathcal{C}'|}{M' \log_2 q'} = \frac{\log_2 8}{5 \log_2 4} = 0.3.$$

Because  $|\mathcal{C}| = |\mathcal{C}'| = 8$ , rate loss, when an additional  $\delta = 1$  column is appended to the source PC, is computed as

$$R_{\text{loss}} = \frac{R - R'}{R} \times 100\% = \frac{0.375 - 0.3}{0.375} \times 100\% = 20\%. \quad (25)$$

Also, according to (21), the lower and upper bounds for the spectrum notches of the extended PC are given by

$$\frac{1}{20} \leq \vartheta|_{f=k/2} \leq \frac{5}{4}. \quad (26)$$

The variation  $\vartheta$  at frequency sub-multiples of  $f = k/G$ , where  $k = 0, 1, 2$  and  $G = 2$ , is calculated from (19) as

$$\vartheta|_{f=0} = \vartheta|_{f=1/2} = \vartheta|_{f=1} = 0.25, \quad (27)$$

which falls within the bounds calculated in (26).

The spectrums of these codebooks (i.e.,  $\mathcal{C}(4,2)$  and  $\mathcal{C}'(5,2,4)$ ) are as shown in Fig. 5. Based on our prediction, the two codebooks exhibit spectral shaping characteristics, with their notches appearing at  $f = 0, 1/2$  and 1. More so, the notches of the extended PC appear at the same values computed in (27).

To demonstrate the significance of  $h$ , which defines the column position of the appended symbols, the target  $\mathcal{C}'(5,2,0)$  codebook was derived (see Table I) by appending the symbols at  $h = 0$  (i.e., the first column). The same variation  $\vartheta$  at  $f = k/G$  is observed as in the case of  $\mathcal{C}'(5,2,4)$  (see Fig. 5).

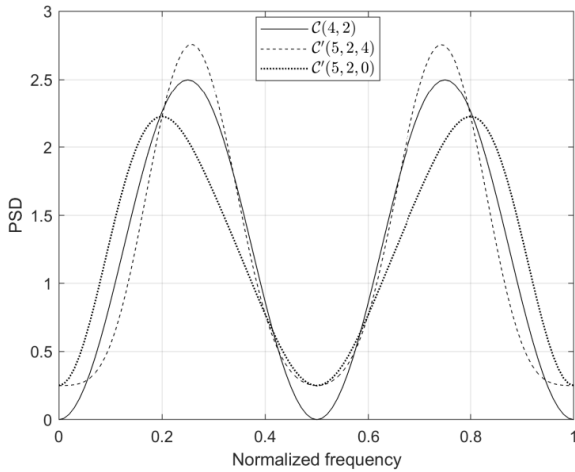


Fig. 5. The spectrums OF AN  $M = 4, G = 2$  spectral null PC and corresponding extended PCs.

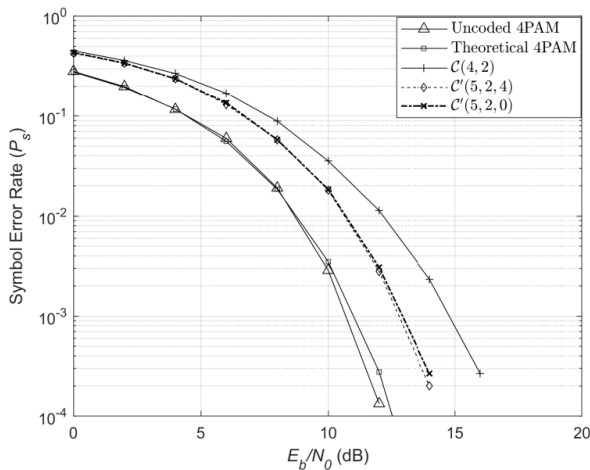


Fig. 6. The performances of an  $M = 4, G = 2$  spectral null PC and corresponding extended PCs under AWGN channel.

The results obtained by simulating the three schemes in Table I, under additive white Gaussian noise (AWGN), are presented in Fig. 6. An uncoded 4PAM scheme was simulated alongside the schemes, so as to validate the correctness of the simulations. Since both schemes  $\mathcal{C}'(5,2,4)$  and  $\mathcal{C}'(5,2,0)$  have higher error-detecting capabilities, they exhibit a better and overlapping performance than the  $\mathcal{C}(4,2)$  scheme.

Next, we consider a source  $\mathcal{C}(6,2)$  spectral null PC, from which  $\mathcal{C}'(7,3,6)$  and  $\mathcal{C}'(7,3,0)$  extended PCs are derived, as shown in Table II.

TABLE II:  $M = 6, G = 3$  SPECTRAL NULL PC AND CORRESPONDING TARGET EXTENDED PC WITH SPECTRAL SHAPING

$\mathcal{C}(6, 3)$ $d_{\min} = 2$ $q = 6$	$\mathcal{C}'(7, 3, 6)$ $d'_{\min} = 3$ $q' = 6$	$\mathcal{C}'(7, 3, 0)$ $d'_{\min} = 3$ $q' = 6$
0 1 2 5 4 3	0 1 2 5 4 3 0	0 0 1 2 5 4 3
1 0 2 4 5 3	1 0 2 4 5 3 1	1 1 0 2 4 5 3
5 1 2 0 4 3	5 1 2 0 4 3 2	2 5 1 2 0 4 3
1 5 2 4 0 3	1 5 2 4 0 3 5	5 1 5 2 4 0 3
0 4 2 5 1 3	0 4 2 5 1 3 4	4 0 4 2 5 1 3
4 0 2 1 5 3	4 0 2 1 5 3 3	3 4 0 2 1 5 3
5 4 2 0 1 3	5 4 2 0 1 3 1	1 5 4 2 0 1 3
4 5 2 1 0 3	4 5 2 1 0 3 0	0 4 5 2 1 0 3
0 1 3 5 4 2	0 1 3 5 4 2 4	4 0 1 3 5 4 2
1 0 3 4 5 2	1 0 3 4 5 2 5	5 1 0 3 4 5 2
5 1 3 0 4 2	5 1 3 0 4 2 3	3 5 1 3 0 4 2
1 5 3 4 0 2	1 5 3 4 0 2 0	0 1 5 3 4 0 2
0 4 3 5 1 2	0 4 3 5 1 2 1	1 0 4 3 5 1 2
4 0 3 1 5 2	4 0 3 1 5 2 2	2 4 0 3 1 5 2
5 4 3 0 1 2	5 4 3 0 1 2 5	5 5 4 3 0 1 2
4 5 3 1 0 2	4 5 3 1 0 2 4	4 4 5 3 1 0 2

The spectral null source PC in Table II is able to detect  $t = 1$  symbol error, while the target PCs  $\mathcal{C}'(7,3,6)$  and  $\mathcal{C}'(7,3,0)$  are able to detect two symbol errors.

The rates of the source and target codebooks are  $R = 0.2579$  and  $R' = 0.2211$ , as computed from (3). The two extended PCs have the same rate since they have the same  $|\mathcal{C}'|$  and  $M'$ . An  $R_{\text{loss}} = 14\%$  rate loss is experienced due to the addition of  $\delta = 1$  column to the source PC.

Also, the two extended PCs have the same lower and upper bounds for the spectrum notches, and these are respectively  $1/28$  and  $7/4$ , as calculated from (21). The variation  $\vartheta$  at frequency sub-multiples of  $f = k/G$ , where  $k = 0, 1, 2, 3$  and  $G = 3$ , is  $0.4643$ , as calculated from (19). This falls within the calculated bounds.

Fig. 7 shows the spectrums of the  $\mathcal{C}(6,3)$ ,  $\mathcal{C}'(7,3,6)$  and  $\mathcal{C}'(7,3,0)$  codebooks. As predicted, the spectral null PC and the two extended PCs exhibit spectral shaping characteristics, having notches at  $f = 0, 1/3, 2/3$  and 1. The notches of the extended PCs also appear at the same computed values of  $\vartheta = 0.4643$ .



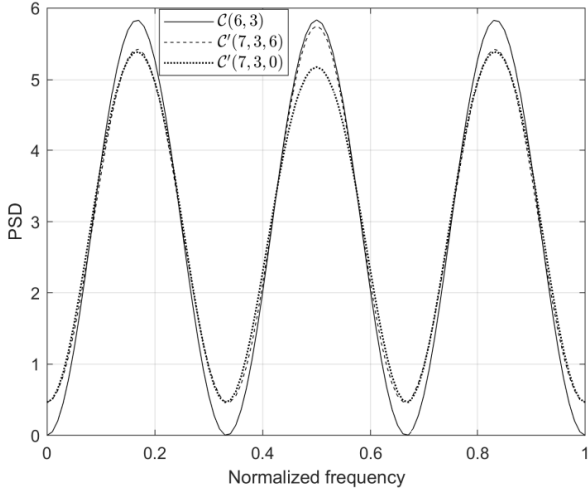


Fig. 7. The spectrums of an  $M = 6, G = 3$  spectral null PC and corresponding extended PCs.

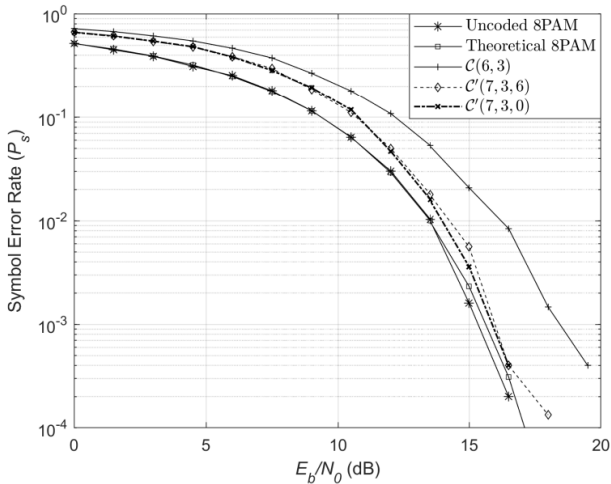


Fig. 8. The performances of an  $M = 6, G = 3$  spectral null PC and corresponding extended PCs under AWGN channel.

As their error-detecting capabilities show, Schemes  $C'(7,3,6)$  and  $C'(7,3,0)$  are able to outperform the  $C(6,3)$  PC scheme under an AWGN channel, as shown in Fig. 8. An uncoded 8PAM scheme was also used to validate the simulations.

For  $M = 8$ , we shall evaluate a  $C(8,4)$  spectral null PC with  $|C| = 32$ , from which  $C'(9,4,8)$  and  $C'(9,4,0)$  are generated (see Table III). Using (3), the rates are computed as  $R = 0.1667$  (for the source PC) and  $R' = 0.1481$  (for the two extended PCs) with  $R_{\text{loss}} = 11\%$ .

The lower and upper bounds for the spectrum notches of the two extended PCs are  $1/36$  and  $9/4$  respectively, and their  $\vartheta$  at  $f = k/G$ , where  $k = 0, 1, 2, 3, 4$  and  $G = 4$ , is calculated to be 0.5833.

Fig. 9 shows the spectrums of the  $C(8,4)$ ,  $C'(9,4,8)$  and  $C'(9,4,0)$  codebooks. The spectral null PC and the two extended PCs are also seen to exhibit spectral shaping characteristics, with their notches appearing at  $f = 0, 1/4, 1/2, 3/4$  and 1.

TABLE III:  $M = 8, G = 4$  SPECTRAL NULL PC AND CORRESPONDING TARGET EXTENDED PC WITH SPECTRAL SHAPING

$C(8,4)$ $d_{\min} = 2$ $q = 8$	$C'(9,4,8)$ $d'_{\min} = 3$ $q' = 8$	$C'(9,4,0)$ $d'_{\min} = 3$ $q' = 8$
0 1 2 3 7 6 5 4	0 1 2 3 7 6 5 4 2	2 0 1 2 3 7 6 5 4
1 0 2 3 6 7 5 4	1 0 2 3 6 7 5 4 3	3 1 0 2 3 6 7 5 4
0 1 2 4 7 6 5 3	0 1 2 4 7 6 5 3 5	5 0 1 2 4 7 6 5 3
1 0 2 4 6 7 5 3	1 0 2 4 6 7 5 3 1	1 1 0 2 4 6 7 5 3
0 1 5 3 7 6 2 4	0 1 5 3 7 6 2 4 7	7 0 1 5 3 7 6 2 4
1 0 5 3 6 7 2 4	1 0 5 3 6 7 2 4 6	6 1 0 5 3 6 7 2 4
0 1 5 4 7 6 2 3	0 1 5 4 7 6 2 3 4	4 0 1 5 4 7 6 2 3
1 0 5 4 6 7 2 3	1 0 5 4 6 7 2 3 5	5 1 0 5 4 6 7 2 3
0 6 2 3 7 1 5 4	0 6 2 3 7 1 5 4 1	1 0 6 2 3 7 1 5 4
6 0 2 3 1 7 5 4	6 0 2 3 1 7 5 4 0	0 6 0 2 3 1 7 5 4
0 6 2 4 7 1 5 3	0 6 2 4 7 1 5 3 4	4 0 6 2 4 7 1 5 3
6 0 2 4 1 7 5 3	6 0 2 4 1 7 5 3 3	3 6 0 2 4 1 7 5 3
0 6 5 3 7 1 2 4	0 6 5 3 7 1 2 4 6	6 0 6 5 3 7 1 2 4
6 0 5 3 1 7 2 4	6 0 5 3 1 7 2 4 7	7 6 0 5 3 1 7 2 4
0 6 5 4 7 1 2 3	0 6 5 4 7 1 2 3 5	5 0 6 5 4 7 1 2 3
6 0 5 4 1 7 2 3	6 0 5 4 1 7 2 3 4	4 6 0 5 4 1 7 2 3
7 1 2 3 0 6 5 4	7 1 2 3 0 6 5 4 7	7 7 1 2 3 0 6 5 4
1 7 2 3 6 0 5 4	1 7 2 3 6 0 5 4 6	6 1 7 2 3 6 0 5 4
7 1 2 4 0 6 5 3	7 1 2 4 0 6 5 3 4	4 7 1 2 4 0 6 5 3
1 7 2 4 6 0 5 3	1 7 2 4 6 0 5 3 5	5 1 7 2 4 6 0 5 3
7 1 5 3 0 6 2 4	7 1 5 3 0 6 2 4 0	0 7 1 5 3 0 6 2 4
1 7 5 3 6 0 2 4	1 7 5 3 6 0 2 4 1	1 1 7 5 3 6 0 2 4
7 1 5 4 0 6 2 3	7 1 5 4 0 6 2 3 3	3 7 1 5 4 0 6 2 3
1 7 5 4 6 0 2 3	1 7 5 4 6 0 2 3 2	2 1 7 5 4 6 0 2 3
7 6 2 3 0 1 5 4	7 6 2 3 0 1 5 4 6	6 7 6 2 3 0 1 5 4
6 7 2 3 1 0 5 4	6 7 2 3 1 0 5 4 7	7 6 7 2 3 1 0 5 4
7 6 2 4 0 1 5 3	7 6 2 4 0 1 5 3 2	2 7 6 2 4 0 1 5 3
6 7 2 4 1 0 5 3	6 7 2 4 1 0 5 3 2	2 6 7 2 4 1 0 5 3
7 6 5 3 0 1 2 4	7 6 5 3 0 1 2 4 1	1 7 6 5 3 0 1 2 4
6 7 5 3 1 0 2 4	6 7 5 3 1 0 2 4 0	0 6 7 5 3 1 0 2 4
7 6 5 4 0 1 2 3	7 6 5 4 0 1 2 3 0	0 7 6 5 4 0 1 2 3
6 7 5 4 1 0 2 3	6 7 5 4 1 0 2 3 3	3 6 7 5 4 1 0 2 3

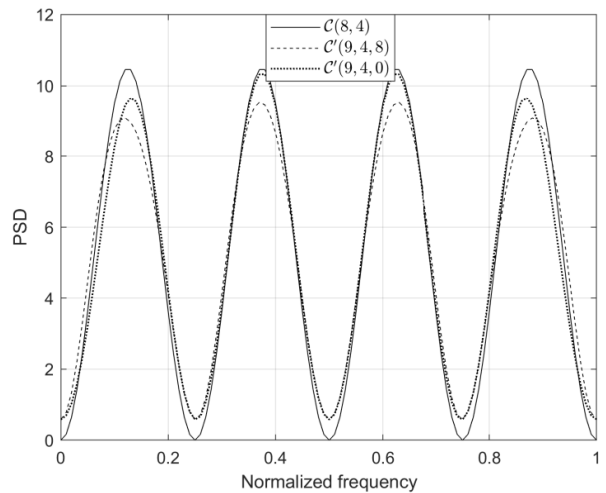


Fig. 9. The spectrums of an  $M = 8, G = 4$  spectral null PC and corresponding extended PCs.

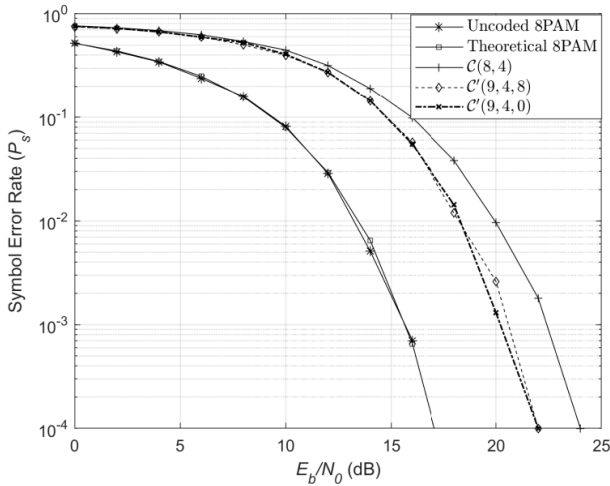


Fig. 10. The performances of an  $M = 8$ ,  $G = 4$  spectral null PC and corresponding extended PCs under AWGN channel.

Similarly, Schemes  $C'(9,4,8)$  and  $C'(9,4,0)$  also outperform the  $C(8,4)$  PC scheme under an AWGN channel, as shown in Fig. 10.

For all the scenarios considered, we see that the rate loss reduces as  $M$  increases. We also see that the same variation  $\vartheta$  is experienced if the appended column is appended at the first or the last column of the source PC. Since the resulting extended PC still retains the spectral shaping capability of the source PC, it therefore offers an advantage of achieving increased minimum Hamming distance, which is a good attribute needed for error correction.

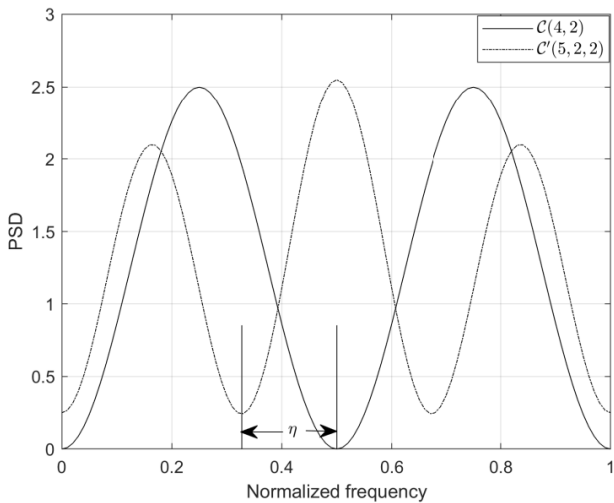


Fig. 11. Illustrating spectral notch drift for a case where  $h = 2$ .

It should be noted that the index  $h$  of the appended columns in this work has been limited to  $h = 0$  and  $h = M - 1$ . Cases where  $1 \leq h < M - 1$  constitute a special case. For instance, if the appended column for the cases considered in Table I is  $h = 2$  (i.e.,  $C'(5,2,2)$ ), the resulting spectrum constitutes a spectrum drift which is a function of  $\eta$ . Here,  $\eta$  is the amount by which the spectrum notches deviate from the expected positions. The spectrums of  $C(4,2)$  and  $C'(5,2,2)$  are as shown in Fig. 11. The spectrum notches of  $C'(5,2,2)$  are seen to be at sub-

multiples of frequency  $1/(G - \eta)$ . Details on such cases are left for future work.

## VI. CONCLUSION

In this paper, we have reported a special form of non-binary coding scheme, which is capable of spectrum shaping at certain frequency sub-multiples. This involves a systematic extension of the codeword length of an existing spectral null permutation codebook, to achieve a new codebook with improved minimum Hamming distance, which in turn assists in better error correction, without losing the spectral shaping capability of the source codebook. The proposed extended spectral shaping permutation coding scheme is a good candidate for satisfying specific transmission requirements, and can, as well, be useful in multiplexing, storage system design or digital recording. More so, since this is used in conjunction with PAM systems, it may be employed in applications such as LAN systems and fiber channels. It should, however be noted that most of the analyses carried out were limited to cases where  $\delta = 1$ . Future work to be done shall consider cases where  $\delta > 1$ .

## CONFLICT OF INTEREST

The authors declare no conflict of interest.

## AUTHOR CONTRIBUTIONS

K. Ogunyanda conducted the research and wrote the paper. T.G. Swart and O.O. Ogunyanda analyzed the results; all authors had approved the final revision before submission.

## REFERENCES

- [1] K. Ouahada, T. G. Swart, and H. C. Ferreira, "Permutation sequences and coded PAM signals with spectral nulls at rational submultiples of the symbol frequency," *Cryptography and Commun.*, vol. 3, no. 2, pp. 87–108, Jun. 2011.
- [2] O. O. Ogunyanda, T. Shongwe, and T. G. Swart, "Injection coding with spectral shaping capability," *IEEE Access*, vol. 9, pp. 19939–19946, 2021.
- [3] O. O. Ogunyanda, T. Shongwe, and T. G. Swart, "Injection coding with spectral nulls for even codeword lengths," *J. Commun.*, vol. 16, no. 5, pp. 167–174, 2021.
- [4] E. Gorog, "Redundant alphabets with desirable frequency spectrum properties," *IBM J. Res. Develop.*, vol. 12, no. 3, pp. 234–241, May 1968.
- [5] K. Ouahada, T. G. Swart, H. C. Ferreira, and L. Cheng, "Spectral shaping technique for permutation distance-preserving mapping codes," in *Proc. IEEE Inf. Theory Workshop*, 2007, pp. 36–41.
- [6] K. Ouahada, "Spectral shaping and distance mappings with permutation sequences," Ph.D. thesis, Univ. Johannesburg, Johannesburg, South Africa, 2009.



- [7] H. C. Ferreira, H. M. Grove, O. Hooijen, and A. J. H. Vinck, "Power line communication," *Wiley Encyclopedia of Electrical and Electronics Engineering*, 2001.
- [8] H. C. Ferreira, D. A. Wright, and A. Nel, "Hamming distance preserving mappings and trellis codes with constrained binary symbols," *IEEE Trans. Inf. Theory*, vol. 35, pp. 1098–1103, Nov. 1989.
- [9] J. I. Hall, "Notes on coding theory," Lecture notes. [Online]. Available: <https://users.math.msu.edu/users/halljo/classes/codenotes/mod.pdf>
- [10] S. Bereg, L. Morales, and I. H. Sudborough, "Extending permutation arrays: Improving MOLS bounds," *Des. Codes Cryptogr.*, vol. 83, pp. 661–683, 2017.
- [11] S. Bereg, L. G. Mojica, L. Morales, and H. Sudborough, "Constructing permutation arrays using partition and extension," *Designs, Codes and Cryptography*, no. 2, pp. 311–339, Oct. 2019.
- [12] S. Bereg, L. G. Mojica, L. Morales, and H. Sudborough, "Parallel partition and extension," in *Proc. 51st Annual Conference on Information Sciences and Systems (CISS)*, 2017, pp. 1–6.
- [13] L. G. Mojica, "Permutation arrays with large Hamming distance," Ph.D. dissertation, Univ. Texas, Dallas, 2017.
- [14] J. Wang, P. Liu, J. Kong, and R. Ying, "Split table extension: A low complexity LVQ extension scheme in low bitrate audio coding," *IEEE Signal Process. Lett.*, vol. 17, no. 1, pp. 59–62, 2010.
- [15] R. Sugiura, Y. Kamamoto, N. Harada, and T. Moriya, "Optimal Golomb-Rice code extension for lossless coding of low entropy exponentially distributed sources," *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 3153–3161, 2018.
- [16] H. C. Ferreira, A. J. H. Vinck, T. G. Swart, and I. de Beer, "Permutation trellis codes," *IEEE Trans. Commun.*, vol. 53, pp. 1782–1789, Nov. 2005.
- [17] H. C. Ferreira and A. J. H. Vinck, "Interference cancellation with permutation trellis codes," in *Proc. Vehicular Technol. Conf., 2000. IEEE-VTS Fall VTC 2000. 52nd*, vol. 5, 2000, pp. 2401–2407.
- [18] T. G. Swart, I. D. Beer, H. C. Ferreira, and A. J. H. Vinck, "Simulation results for permutation trellis codes using M-ary FSK," in *IEEE Int. Symp. Power Line Commun. Appl. (ISPLC)*, Vancouver, CA, Apr. 2005, pp. 317–321.
- [19] T. G. Swart, I. D. Beer, and H. C. Ferreira, "On the distance optimality of permutation mappings," in *Proc. IEEE Int. Symp. on Inf. Theory*, Adelaide, Australia, Sept. 2005, pp. 1068–1072.
- [20] S. Haykin, *Communication Systems*, 4th ed., Wiley, 2001.
- [21] K. A. S. Immink, *Codes for Mass Data Storage Systems*, The Netherlands: Shannon Foundation Publishers, 1999.
- [22] K. A. S. Immink, *Coding Techniques for Digital Recorders*. UK: Prentice Hall International, 1991.

Copyright © 2022 by the authors. This is an open access article distributed under the Creative Commons Attribution License ([CC BY-NC-ND 4.0](https://creativecommons.org/licenses/by-nc-nd/4.0/)), which permits use, distribution and reproduction in any medium, provided that the article is properly cited, the use is non-commercial and no modifications or adaptations are made.



**Kehinde Ogunyanda** received his B. Eng. Electrical Engineering from University of Ilorin, Nigeria, in 2006, M. Tech. degree in Electrical Engineering from Cape Peninsula University of Technology, South Africa, in 2013, M.Sc. degree in Electronic and Electrical Systems from Ecole Supérieure d'Ingenieurs en Electronique et Electrotechnique, Paris, in 2013, and D. Phil Electrical and Electronic Engineering from University of Johannesburg, South Africa in 2018. He is currently a Postdoctoral Research Fellow at the University of Johannesburg, South Africa. To date, he has about 8 publications to his name. His research interests include error-correction coding, digital communications, software defined radio technology, wireless local area network technology and power-line communications.



**Theo G. Swart** received the B.Eng. and M.Eng. degrees (both cum laude) in electrical and electronic engineering from the Rand Afrikaans University, South Africa, in 1999 and 2001, respectively, and the D.Eng. degree from the University of Johannesburg, South Africa, in 2006. He is an associate professor in the Department of Electrical and Electronic Engineering Science and the director of the Center for Telecommunications at the University of Johannesburg. He was previously the chair of the IEEE South Africa Chapter on Information Theory. He has served on several technical program committees for IEEE conferences and regularly serves as a reviewer for IEEE journals and conferences. He is also a specialist editor in Communications and Signal Processing for the SAIEE Africa Research Journal. His research interests include digital communications, error-correction coding, constrained coding and power-line communications. To date he has over 25 journal articles and over 65 conference articles to his name, and has been the co-editor of and contributor to two editions of a comprehensive book on power-line communications.



**Opeyemi O. Ogunyanda** received the B.Tech. degree in computer engineering from Ladoke Akintola University of Technology (LAUTECH), Ogbomoso, Nigeria, in 2011 and the M.Eng. degree in electrical and electronic engineering from Federal University of Technology Akure, Akure, Nigeria, in 2016. She has just completed a Ph.D. degree in electrical and electronic engineering at University of Johannesburg, Johannesburg, South Africa. From 2009 to 2010, she was the secretary of the editorial committee at Nigerian Association of Computer Science Students, LAUTECH chapter, Ogbomoso, Nigeria. Between 2017 and 2021, she has authored 4 articles. Her research interests include spectrum shaping using channel coding such as permutation coding and injection coding schemes in digital communications applications.