Bounds on the Size of Permutation Codes with the Kendall τ-Metric

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***Abstract*—The rank modulation scheme has been proposed for efficient writing and storing data in non-volatile memory storage. Error-correction in the rank modulation scheme is done by considering permutation codes. In this paper we consider codes in the set of all permutations on n elements, S_n, using the Kendall τ-metric. The main goal of this paper is to derive new bounds on the size of such codes. For this purpose we also consider perfect codes, diameter perfect codes, and the size of optimal anticodes in the Kendall τ-metric, structures which have their own considerable interest. We prove that there are no perfect single-error-correcting codes in S_n, where $n > 4$ is a prime or $4 \leq n \leq 10$. We present lower bounds on the size of optimal anticodes with odd diameter. As a consequence we obtain a new upper bound on the size of codes in S_n with even minimum Kendall τ-distance. We present larger single-error-correcting codes than the known ones in S_5 and S_7.**

***Index Terms*—Anticodes, bounds, flash memory, Kendall τ-metric, perfect codes, permutations**

I. Introduction

FLASH memory is a non-volatile technology that is both electrically programmable and electrically erasable. It incorporates a set of cells maintained at a set of levels of charge to encode information. While raising the charge level of a cell is an easy operation, reducing the charge level requires the erasure of the whole block to which the cell belongs. For this reason charge is injected into the cell over several iterations. Such programming is slow and can cause errors since cells may be injected with extra unwanted charge. Other common errors in flash memory cells are due to charge leakage and reading disturbance that may cause charge to move from one cell to its adjacent cells. In order to overcome these problems, the novel framework of *rank modulation codes* was introduced in [20]. In this setup the information is carried by the relative ranking of the cells charge levels and not by the absolute values of the charge levels. This allows for more efficient programming of cells, and coding by the ranking of the cells' levels is more robust to charge leakage than coding by their actual values. In this model codes are subsets of S_n, the set of all permutations on n elements, where each permutation corresponds to a ranking of n cells' levels. Permutation codes were mainly studied in this context using three metrics, the infinity metric, the Ulam metric, and the Kendall τ-metric. Codes in S_n under the infinity metric were considered in [24], [36], [38], [40]. Anticodes in S_n under the infinity metric were considered in [23], [37], [39]. Codes in S_n under the Ulam metric were considered in [16]. Permutation codes with other metrics were considered in many papers. A survey on metrics related to permutations is given in [11].

In this paper we consider codes using the Kendall τ-metric [22]. Under the Kendall τ-metric, codes in S_n with minimum distance d should correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors that are caused by small charge leakage and read disturbance. For large charge leakage and read disturbance the Ulam metric is used [16]. Let $P(n,d)$ denote the size of the largest code in S_n with minimum Kendall τ-distance d. A comprehensive work on error-correcting codes in S_n using the Kendall τ-metric and bounds on $P(n,d)$ were considered in [21]. In that paper there is also a construction of single-error-correcting codes using codes in the Lee metric. This method was generalized in [3] for the construction of t-error-correcting codes that are of optimal size up to a constant factor, where t is fixed. More constructions of error-correcting codes were given in [28]. Systematic single-error-correcting codes in S_n of size $(n-2)!$ were constructed in [41], [42]. The constructed codes are of optimal size, assuming that perfect single-error-correcting codes do not exist. But, only the nonexistence of perfect single-error-correcting codes for $n = 4$ was proved. Systematic t-error-correcting codes were studied in [6], [41], [42]. Linear programming and semi-definite programming on permutation codes with the Kendall τ-metric were considered in [26]. Unfortunately, no bounds better than the sphere packing bound were found by these methods.

The main goal of this paper is to provide new bounds on the size of permutation codes in the Kendall τ-metric.

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As part of this goal we will prove the nonexistence of perfect single-error-correcting codes in S_n if n is a prime. Although this improves the related upper bound on $P(n,3)$ only by one, such a result is of interest for itself. This is one of the two main results of this paper. The second main result is a new upper bound on the size of permutation codes in the Kendall τ-metric, where the minimum distance is even. This bound is obtained by introducing the notion of anticodes in the Kendall τ-metric and proving a related code-anticode theorem. Finally, we present two codes with minimum distance 3 in S_5 and S_7, which are considerably larger than the previous known codes. These codes are of special interest since the rank modulation scheme is more likely to be applicable for small values of n.

The rest of this work is organized as follows. In Section II we define the basic concepts for the Kendall τ-metric and for perfect codes. In Section III we prove the nonexistence of a perfect single-error-correcting code in S_n, using the Kendall τ-metric, where $n > 4$ is a prime or $4 \leq n \leq 10$. This is the first known result in this direction and it shows that the sphere packing upper bound can not be attained in these cases. In Section IV we establish the Delsarte's code-anticode bound for the Kendall τ-metric and examine diameter perfect codes in S_n for this metric. We find the sizes of optimal anticodes in S_n with diameter 2 and diameter 3 and consider the size of optimal anticodes for larger diameters as well. Trivial diameter perfect codes are considered in some of these cases. We combine these results with the code-anticode bound to improve the known upper bound on the size of a code in S_n for even minimum distances. In Section V we consider lower bounds on the size of permutation codes in the Kendall τ-metric for small values of n. We search for such codes by forcing a structure and a certain automorphism group on the codes. Two large single-error-correcting codes for $n = 5$ and $n = 7$ are constructed in this way and yield an improvement on the related lower bounds. We conclude in Section VI, where we also present some questions for future research.

II. Basic Concepts

Let S_n be the set of all permutations on the set of n elements $[n] \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$. We denote a permutation $\sigma \in S_n$ by $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(n)]$. For two permutations $\sigma, \pi \in S_n$, their multiplication $\pi \circ \sigma$ is defined as the composition of σ on π, namely, $\pi \circ \sigma(i) = \sigma(\pi(i))$, for all $1 \leq i \leq n$. Under this operation, the set S_n is a noncommutative group, known as the symmetric group of order $n!$. We denote by $\varepsilon \stackrel{\text{def}}{=} [1, 2, \ldots, n]$ the identity permutation of S_n. Given a permutation $\sigma \in S_n$, an *adjacent transposition*, $(i, i+1)$, for some $1 \leq i \leq n-1$, is an exchange of the two adjacent elements $\sigma(i)$ and $\sigma(i+1)$ in σ. The result is the permutation $\pi = [\sigma(1), \ldots, \sigma(i-1), \sigma(i+1), \sigma(i), \sigma(i+2), \ldots, \sigma(n)]$. Observe that the notation $(i, i+1)$ is also used for the cycle decomposition of the permutation $[1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n]$ and the permutation π can also be written as $\pi = (i, i+1) \circ \sigma$. In other words, left multiplication by $(i, i+1)$ exchanges the elements in positions $i, i+1$. Right multiplication by $(i, i+1)$ exchanges the elements $i, i+1$. Two adjacent transpositions $(i, i+1)$ and $(j, j+1)$ are called *disjoint* if either $i+1 < j$ or $j+1 < i$. For two permutations $\sigma, \pi \in S_n$, the Kendall τ-distance between σ and π, $d_K(\sigma, \pi)$, is defined as the minimum number of adjacent transpositions needed to transform σ into π [22]. For $\sigma \in S_n$, the Kendall τ-weight of σ, $w_K(\sigma)$, is defined as the Kendall τ-distance between σ and the identity permutation ε. The following expression for $d_K(\sigma, \pi)$ is well known [21], [25].

$$d_K(\sigma, \pi) = |\{(i, j) \,:\, \sigma^{-1}(i) < \sigma^{-1}(j) \wedge \pi^{-1}(i) > \pi^{-1}(j)\}|. \tag{1}$$

For a permutation $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(n)] \in S_n$, the *reverse* of σ is the permutation $\sigma^r \stackrel{\text{def}}{=} [\sigma(n), \sigma(n-1), \ldots, \sigma(2), \sigma(1)]$. It follows from equation (1) that for every $\sigma, \pi \in S_n$, $d_K(\sigma, \pi) \leq \binom{n}{2}$ and $d_K(\sigma, \pi) = \binom{n}{2}$ if and only if $\pi = \sigma^r$. The following lemma is an immediate consequence from the expression to compute the Kendall τ-distance given in (1).

Lemma 1. *For every* $\sigma, \pi \in S_n$,

$$d_K(\sigma, \pi) + d_K(\sigma^r, \pi) = d_K(\sigma, \sigma^r) = \binom{n}{2}.$$

The Kendall τ-metric is right invariant [7], [11], i.e. for every three permutations $\sigma, \pi, \rho \in S_n$ we have $d_K(\sigma, \pi) = d_K(\sigma \circ \rho, \pi \circ \rho)$. Note, that the Kendall τ-metric is not left invariant. The Kendall τ-metric on S_n is graphic, i.e. for every two permutations $\sigma, \pi \in S_n$ their Kendall τ-distance is equal to the length of the shortest path between σ and π in the graph G_n, whose vertex set is the set S_n, and two vertices are connected by an edge if and only if their Kendall τ-distance is one.

A distance measure $d(\cdot, \cdot)$ over a space $\mathcal{V}$, is called *bipartite* if every three elements $x, y, z \in \mathcal{V}$ satisfy the equality $d(x, y) + d(y, z) \equiv d(x, z) \pmod 2$, i.e. the related graph is bipartite. The Kendall τ-metric on S_n is bipartite as stated in the next lemma.

Lemma 2. *The Kendall τ-metric over S_n is bipartite.*

Proof: Just note that by (1) two permutations which differ in exactly one adjacent transposition have different weights modulo 2. This implies that the related graph G_n and the Kendall τ-metric are bipartite. ■

Corollary 1. *If σ and π are two permutations in S_n then $w_K(\sigma) + w_K(\pi) \equiv w_K(\sigma \circ \pi)$ (mod 2).*

Proof: Since the Kendall τ-metric is right invariant, it follows that $w_K(\pi) = d_K(\pi, \epsilon) = d_K(\epsilon, \pi^{-1}) = w_K(\pi^{-1})$. Hence, by the definition of the Kendall τ-weight and by Lemma 2, we have that

$$w_K(\sigma) + w_K(\pi) = w_K(\sigma) + w_K(\pi^{-1})$$

$$= d_K(\sigma, \epsilon) + d_K(\pi^{-1}, \epsilon) \equiv d_K(\sigma, \pi^{-1}) \ (\text{mod } 2) \ . \quad (2)$$

Since the Kendall τ-metric is right invariant, it follows that

$$d_K(\sigma, \pi^{-1}) = d_K(\sigma \circ \pi, \epsilon) = w_K(\sigma \circ \pi) \qquad (3)$$

Thus, by (2) and (3), we have that $w_K(\sigma) + w_K(\pi) \equiv w_K(\sigma \circ \pi)$ (mod 2). ■

Given a metric space, one can define codes. We say that $\mathcal{C} \subseteq S_n$ has *minimum distance* d if $d_K(\sigma, \pi) \geq d$, for every two distinct permutations $\sigma, \pi \in \mathcal{C}$. For a given space $\mathcal{V}$ with a distance measure $d(\cdot, \cdot)$, a subset C of $\mathcal{V}$ is a *perfect code* with *radius* R if for every element $x \in \mathcal{V}$ there exists exactly one codeword $c \in C$ such that $d(x, c) \leq R$. For a point $x \in \mathcal{V}$, the *ball* of radius R centered at x, $B(x, R)$, is defined by $B(x, R) \stackrel{\text{def}}{=} \{y \in \mathcal{V} \ : \ d(x, y) \leq R\}$. In the Kendall τ-metric the size of a ball does not depend on the center of the ball. This is a consequence of the fact that the Kendall τ-distance is right invariant. It is readily verified that

Theorem 1. *Let $\mathcal{V}$ be a space with a distance measure $d(\cdot, \cdot)$. For a code $C \subseteq \mathcal{V}$ with minimum distance $2R+1$ and a ball B with radius R we have $|C| \cdot |B| \leq |\mathcal{V}|$, where $|S|$ is the size of the set S.*

Theorem 1 is known as the *sphere packing bound* (even so it is really a ball packing bound). In a code C which attains this bound, i.e. $|C| \cdot |B| = |\mathcal{V}|$, the balls with radius R around the codewords of C form a partition of $\mathcal{V}$. Such a code is a perfect code. A perfect code with radius R is also called a *perfect R-error-correcting code*.

Perfect codes is one of the most fascinating topics in coding theory. These codes were mainly considered for the Hamming scheme, e.g. [15], [29], [31]–[33]. They were also considered for other schemes such as the Johnson scheme, e.g. [12], [14], [35], the Grassmann scheme [8], [27], and to a larger extent also in the Lee and the Manhattan metrics, e.g. [13], [17], [18], [34]. Note, that the minimum distance of a perfect code is always an odd integer. A more general concept in which codes can have even minimum distances as well, is a diameter perfect code [1]. This concept is based on Delsarte's code-anticode bound [10] for distance regular graphs. Since the Kendall τ-metric over S_n does not induce a distance regular graph, Delsarte's theorem may not apply for this metric. However, an alternative proof shows that such type of a bound is also valid for the Kendall τ-metric.

III. The Nonexistence of Some Perfect Codes

In this section we prove that there are no single-error-correcting codes in S_n, where n is a prime greater than 4. Similarly, we also show that there are no perfect single-error-correcting codes in S_n, for $4 \leq n \leq 10$.

For each i, $1 \leq i \leq n$, we define $T_{n,i} \stackrel{\text{def}}{=} \{\sigma \ : \ \sigma \in S_n, \ \sigma(i) = 1\}$, i.e. $\sigma \in S_n$ is an element of $T_{n,i}$ if 1 appears in the ith position of σ. Clearly, $|T_{n,i}| = (n-1)!$.

Assume that there exists a perfect single-error-correcting code $\mathcal{C} \subset S_n$. For each i, $1 \leq i \leq n$, let

$$\mathcal{C}_i \stackrel{\text{def}}{=} \mathcal{C} \cap T_{n,i} \qquad \text{and} \qquad x_i \stackrel{\text{def}}{=} |\mathcal{C}_i|.$$

We say that a codeword $\sigma \in \mathcal{C}$ *covers* a permutation $\pi \in S_n$ if $d_K(\sigma, \pi) \leq 1$. Since $\mathcal{C}$ is a perfect single-error-correcting code, it follows that each permutation in $T_{n,1}$ must be at distance at most one from exactly one codeword of $\mathcal{C}$ and this codeword must belong to either $\mathcal{C}_1$ or $\mathcal{C}_2$. Every codeword $\sigma \in \mathcal{C}_1$ covers exactly $n-1$ permutations in $T_{n,1}$. It covers itself and the $n-2$ permutations in $T_{n,1}$ obtained from σ by exactly one adjacent transposition $(i, i+1)$, $1 < i < n$. Each codeword $\sigma \in \mathcal{C}_2$ covers exactly one permutation $\pi \in T_{n,1}$, $\pi = (1, 2) \circ \sigma$. Therefore, we have that

$$(n-1)x_1 + x_2 = (n-1)! \ . \qquad (4)$$

Similarly, by considering how the permutations of $T_{n,n}$ are covered by the codewords of $\mathcal{C}$, we have that

$$x_{n-1} + (n-1)x_n = (n-1)! \ . \qquad (5)$$

For each i, $2 \leq i \leq n-1$, each permutation in $T_{n,i}$ is covered by exactly one codeword that belongs to either $\mathcal{C}_{i-1}$, $\mathcal{C}_i$, or $\mathcal{C}_{i+1}$. Each codeword $\sigma \in \mathcal{C}_i$ covers exactly $n-2$ permutations in $T_{n,i}$. It covers itself and the $n-3$ permutations in $T_{n,i}$ obtained from σ by exactly one adjacent transposition $(j, j+1)$, where $1 \leq j < i-1$ or $i < j < n$. Each codeword in $\mathcal{C}_{i-1} \cup \mathcal{C}_{i+1}$ covers exactly one permutation from $T_{n,i}$. Therefore, for each i, $2 \leq i \leq n-1$, we have that

$$x_{i-1} + (n-2)x_i + x_{i+1} = (n-1)! \ . \qquad (6)$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and let $\mathbf{1}$ denote the all-ones column vector. Equations (4), (5), and (6) can be written in a matrix form as

$$Ax^T = (n-1)! \cdot \mathbf{1}, \tag{7}$$

where $A = (a_{i,j})$ is an $n \times n$ matrix defined by

$$A = \begin{pmatrix} n-1 & 1 & 0 & 0 & \cdots & 0 & 0 & \dots & 0 \\ 1 & n-2 & 1 & 0 & \cdots & 0 & 0 & \dots & 0 \\ 0 & 1 & n-2 & 1 & \cdots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \cdots & 1 & n-2 & 1 & 0 \\ 0 & \dots & 0 & 0 & \cdots & 0 & 1 & n-2 & 1 \\ 0 & \dots & 0 & 0 & \cdots & 0 & 0 & 1 & n-1 \end{pmatrix}.$$

Since the sum of every row in A is equal to n it follows that the linear equation system defined in (7) has a solution $\mathbf{y}^T = \frac{(n-1)!}{n} \cdot \mathbf{1}$. We will show that if $n > 3$ then A is a nonsingular matrix and hence $\mathbf{y}$ is the unique solution of (7), i.e. $\mathbf{x} = \mathbf{y}$. To this end, we need the following theorem known as the Levy-Desplanques Theorem [19, p. 125].

Theorem 2. *Let $B = (b_{i,j})$ be an $n \times n$ matrix. If $|b_{i,i}| > \sum_{j \neq i} |b_{i,j}|$ for all i, $1 \leq i \leq n$, then B is nonsingular.*

For every $n > 4$ we have that for each i, $1 \leq i \leq n$, $a_{i,i} \geq n-2 > 2 \geq \sum_{j \neq i} a_{i,j}$. Hence, by Theorem 2 it follows that A is nonsingular. For $n = 4$ it can be readily verified that the matrix A is nonsingular. As a consequence we have that $\mathbf{x}^T = \frac{(n-1)!}{n} \cdot \mathbf{1}$ for every $n \geq 4$. If $n = 4$ or n is a prime greater than 4 then $\frac{(n-1)!}{n}$ is not an integer and therefore, a perfect single-error-correcting code does not exist, i.e.

Theorem 3. *There is no perfect single-error-correcting code in S_n, where $n > 4$ is a prime or $n = 4$.*

Remark 1. *It was brought to our attention that Theorem 3 is a special case of Theorem 5 in [9]. However, there is a crucial mistake in the proof of this theorem, which cannot be resolved. The proof follows by induction on n, where the induction step is based on a partition of S_n into $\binom{n}{k}$ classes, $2 \leq k \leq n-2$, according to the set of the k first elements in the permutations. It is stated that if $\mathcal{C} \subset S_n$ is a code with minimum distance 3 and $\mathcal{C}$ is contained in one of these classes, then the projection of $\mathcal{C}$ into S_k has also minimum distance 3. This argument is clearly wrong. For example, the code $\{[1,2,3,4,5],[3,1,2,5,4]\}$ has minimum distance 3 and the first three elements in each of its codewords belong to $\{1,2,3\}$. However, its projection into S_3 is the code $\{[1,2,3],[3,1,2]\}$, which has minimum distance 2. A similar example can be found for every $n \geq 4$ and for each $2 \leq k \leq n-2$.*

The following theorem proved in [5] implies that perfect single-error-correcting codes must have a very symmetric and uniform structure. This might be useful to rule out the existence of these codes for other parameters as well. The proof of this theorem is a generalization of the technique used to prove Theorem 3. It is omitted here since the theorem is not used in the sequel.

Theorem 4. *Assume that there exists a perfect single-error-correcting code $\mathcal{C} \subset S_n$, where $n > 11$. If $r < \frac{n}{4}$ then for each sequence of r distinct elements of $[n]$, $i_1, i_2, \ldots, i_r$, and for each set of r positions, $1 \leq j_1 < j_2 < \cdots < j_r \leq n$, there are exactly $\frac{(n-r)!}{n}$ codewords $\sigma \in \mathcal{C}$, such that $\sigma(j_\ell) = i_\ell$, for each ℓ, $1 \leq \ell \leq r$.*

For $n = 6, 8, 9, 10$, we use similar arguments and obtain systems of linear equations. We used a computer to show that these systems have no solutions over the nonnegative integers, and to conclude that perfect single-error-correcting codes in S_n do not exist for these values of n. More details on these cases can be found in Appendix A.

Corollary 2. *$P(n,3) < (n-1)!$ if n is a prime greater than 4 or $4 \leq n \leq 10$.*

Proof: The size of a ball with radius one in S_n, when the Kendall τ-metric is used, is n. Hence, by Theorem 1 and the discussion which follows this theorem we have that, a single-error-correcting code $\mathcal{C} \subset S_n$ is perfect if and only if $|\mathcal{C}| = (n-1)!$. Since such codes do not exist if n is a prime greater than 4 or if $4 \leq n \leq 10$, it follows that $P(n,3) < (n-1)!$. ■

IV. Anticodes and Diameter Perfect Codes

In all the perfect codes of a graphic metric the minimum distance of the code is an odd integer. If the minimum distance of the code C is an even integer then C cannot be a perfect code. The reason is that for any two codewords $c_1, c_2 \in C$ such that $d(c_1, c_2) = 2\delta$, there exists a word x such that $d(x, c_1) = \delta$ and $d(x, c_2) = \delta$. For this case another concept is used, a diameter perfect code, as was defined in [1]. This concept is based on the code-anticode bound presented by Delsarte [10]. An *anticode* $\mathcal{A}$ of *diameter* D in a space $\mathcal{V}$ is a subset of words from $\mathcal{V}$ such that $d(x,y) \leq D$ for all $x, y \in \mathcal{A}$.

Theorem 5. *If a code C, in a space $\mathcal{V}$ of a distance regular graph, has minimum distance d and in an anticode $\mathcal{A}$ of the space $\mathcal{V}$ the maximum distance is $d-1$ then $|C| \cdot |\mathcal{A}| \leq |\mathcal{V}|$.*

Theorem 5 which was proved in [10] is a generalization of Theorem 1 (the sphere packing bound) and it can be applied to the Hamming scheme since the related graph is distance regular (see [4] for the definition of a distance regular graph). It cannot be applied to the Kendall τ-metric since the related graph is not distance regular if $n > 3$. This can be easily verified by considering the three permutations $\varepsilon = [1,2,3,4,5,\ldots,n]$, $\sigma = [3,1,2,4,5,\ldots,n]$, and $\pi = [2,1,4,3,5,\ldots,n]$

in S_n. Clearly, $d_K(\varepsilon,\sigma) = d_K(\varepsilon,\pi) = 2$ and there exists exactly one permutation α for which $d_K(\varepsilon,\alpha) = 1$ and $d_K(\alpha,\sigma) = 1$, while there exist exactly two permutations β, γ for which $d_K(\varepsilon,\beta) = 1$, $d_K(\beta,\pi) = 1$, $d_K(\varepsilon,\gamma) = 1$, and $d_K(\gamma,\pi) = 1$. Fortunately, an alternative proof which was given in [1] and was modified in [13] will work for the Kendall τ-metric.

Theorem 6. *Let $\mathcal{C}_{\mathcal{D}}$ be a code in S_n with Kendall τ-distances between codewords taken from a set $\mathcal{D}$. Let $\mathcal{A} \subset S_n$ and let $\mathcal{C}'_{\mathcal{D}}$ be the largest code in $\mathcal{A}$ with Kendall τ-distances between codewords taken from the set $\mathcal{D}$. Then*

$$\frac{|\mathcal{C}_{\mathcal{D}}|}{n!} \leq \frac{|\mathcal{C}'_{\mathcal{D}}|}{|\mathcal{A}|} \ .$$

Proof: Let $\mathcal{B} \overset{\text{def}}{=} \{(\sigma,\pi) \ : \ \sigma \in \mathcal{C}_{\mathcal{D}}, \ \pi \in S_n, \ \sigma \circ \pi \in \mathcal{A}\}$. For a given codeword $\sigma \in \mathcal{C}_{\mathcal{D}}$ and a word $\alpha \in \mathcal{A}$, there is exactly one element $\pi \in S_n$ such that $\alpha = \sigma \circ \pi$. Therefore, $|\mathcal{B}| = |\mathcal{C}_{\mathcal{D}}| \cdot |\mathcal{A}|$.

Since the Kendall τ-metric is right invariant it follows that for every $\pi \in S_n$, the set $\mathcal{C}_\pi \overset{\text{def}}{=} \{\sigma \circ \pi \ : \ \sigma \in \mathcal{C}_{\mathcal{D}}\}$ has the same Kendall τ-distances as in $\mathcal{C}_{\mathcal{D}}$, i.e. the Kendall τ-distances between codewords of $\mathcal{C}_\pi$ are taken from the set $\mathcal{D}$. Together with the fact that $\mathcal{C}'_{\mathcal{D}}$ is the largest code in $\mathcal{A}$, with Kendall τ-distances between codewords taken from the set $\mathcal{D}$, it follows that for any given word $\pi \in S_n$ the set $\{\sigma \ : \ \sigma \in \mathcal{C}_{\mathcal{D}}, \ \sigma \circ \pi \in \mathcal{A}\}$ has at most $|\mathcal{C}'_{\mathcal{D}}|$ codewords. Hence, $|\mathcal{B}| \leq |\mathcal{C}'_{\mathcal{D}}| \cdot n!$.

Thus, since $|\mathcal{B}| = |\mathcal{C}_{\mathcal{D}}| \cdot |\mathcal{A}|$, we have that $|\mathcal{C}_{\mathcal{D}}| \cdot |\mathcal{A}| \leq |\mathcal{C}'_{\mathcal{D}}| \cdot n!$ and the claim is proved. ∎

Corollary 3. *If a code $\mathcal{C} \subseteq S_n$ has minimum Kendall τ-distance d and in an anticode $\mathcal{A} \subset S_n$ the maximum Kendall τ-distance is $d-1$ then $|\mathcal{C}| \cdot |\mathcal{A}| \leq n!$.*

Proof: Let $\mathcal{D} = \{d, d+1, \ldots, \binom{n}{2}\}$ and let $\mathcal{C}_{\mathcal{D}} \subseteq S_n$ be a code with minimum Kendall τ-distance d. Let $\mathcal{A}$ be a subset of S_n with Kendall τ-distances between words of $\mathcal{A}$ taken from the set $\{1, 2, \ldots, d-1\}$, i.e. $\mathcal{A}$ is an anticode with diameter $d-1$. Clearly, the largest code in $\mathcal{A}$ with Kendall τ-distances from $\mathcal{D}$ has only one codeword. Applying Theorem 6 on $\mathcal{D}$, $\mathcal{C}_{\mathcal{D}}$, and $\mathcal{A}$, implies that $|\mathcal{C}_{\mathcal{D}}| \cdot |\mathcal{A}| \leq n!$. ∎

If there exists a code $\mathcal{C} \subseteq S_n$ with minimum Kendall τ-distance $d = D+1$ and an anticode $\mathcal{A}$ with diameter D such that $|\mathcal{C}| \cdot |\mathcal{A}| = n!$ then $\mathcal{C}$ is called a *D-diameter perfect* code. In this case, $\mathcal{A}$ must be an anticode with maximum distance (diameter) D of the largest possible size, and $\mathcal{A}$ is called an *optimal* anticode of diameter D. If $D = 2R$ and the ball of radius R is an optimal anticode then a D-diameter perfect code is a perfect R-error-correcting code. It is interesting to find the optimal anticodes in S_n and to determine their sizes. Using the sizes of such optimal anticodes we can obtain by Corollary 3 upper bounds on $P(n, 2\delta)$. In the rest of this section we will mostly consider bounds on the size of optimal anticodes and use these bounds to obtain new upper bounds on $P(n, 2\delta)$. The proof of the next theorem is given in Appendix B.

Theorem 7. *Every optimal anticode with diameter 2 (using the Kendall τ-distance) in S_n, $n \geq 5$, is a ball with radius one whose size is n.*

We will now consider lower bounds on the size of optimal anticodes with odd diameter. These bounds will imply new lower bounds on $P(n, 2\delta)$. To this end we will define a double ball of radius R. For a given space $\mathcal{V}$ with a distance measure $d(\cdot,\cdot)$ and for two elements $x, y \in \mathcal{V}$ such that $d(x,y) = 1$, the *double ball* of radius R centered at x and y is defined by $DB(x,y,R) \overset{\text{def}}{=} B(\mathbf{x}, R) \cup B(\mathbf{y}, R)$. Let $B_{n,R}$ be a ball of radius R in S_n. W.l.o.g., we may assume that $B_{n,R} = B(\varepsilon, R)$. For every $n \geq 1$ and $R \geq 0$, we denote by $DB_{n,R}$ the double ball of radius R in S_n centered at the identity permutation ε and the permutation $(1,2)$.

Lemma 3. *Let $\mathcal{V}$ be a space with a distance measure $d(\cdot,\cdot)$. For every $x, y \in \mathcal{V}$ such that $d(x,y) = 1$ we have*

(1) *$DB(x,y,R)$ is an anticode of diameter at most $2R+1$.*
(2) *$|DB(x,y,R)| = |B(x,R)| + |B(y,R)| - |B(x,R) \cap B(y,R)|$.*
(3) *If $d(\cdot,\cdot)$ over $\mathcal{V}$ is bipartite then $B(x,R) \cap B(y,R) = DB(x,y,R-1)$.*

Proof: (1) follows immediately from the triangle inequality and (2) is trivial.

If $z \in B(x,R) \cap B(y,R)$ then $d(x,z) \leq R$ and $d(y,z) \leq R$. Assume that $d(\cdot,\cdot)$ is bipartite, i.e. every three elements $\hat{x}, \hat{y}, \hat{z} \in \mathcal{V}$ satisfies the equation $d(\hat{x},\hat{y}) + d(\hat{y},\hat{z}) \equiv d(\hat{x},\hat{z}) \pmod 2$. If $d(x,z) = d(y,z) = R$ then $d(x,y) + d(y,z) \not\equiv d(x,z) \pmod 2$, a contradiction. Hence, $d(x,z) \leq R-1$ or $d(y,z) \leq R-1$ and therefore, $z \in DB(x,y,R-1)$.

On the other hand, if $z \in DB(x,y,R-1)$ then $d(x,z) \leq R-1$ or $d(y,z) \leq R-1$ and since $d(x,y) = 1$ it follows from the triangle inequality that $d(x,z) \leq R$ and $d(y,z) \leq R$. Therefore, $z \in B(x,R) \cap B(y,R)$.

Thus, $z \in B(x,R) \cap B(y,R)$ if and only if $z \in DB(x,y,R-1)$, i.e. $B(x,R) \cap B(y,R) = DN(x,y,R-1)$. ∎

Corollary 4. $|DB_{n,R}| = 2|B_{n,R}| - |DB_{n,R-1}|$.

Proof: By Lemma 3 (2) we have $|DB_{n,R}| = 2|B_{n,R}| - |B(\varepsilon,R) \cap B((1,2),R)|$. By Lemma 3 (3) we have that $|B(\varepsilon,R) \cap B((1,2),R)| = DB_{n-1,R}$. Thus, $|DB_{n,R}| = 2|B_{n,R}| - |DB_{n,R-1}|$. ∎

Theorem 8. *If $n \geq 4$ then $DB_{n,1}$ is an optimal anticode of diameter 3, whose size is $2(n-1)$.*

Proof: The claim can be easily verified for $n = 4$. By the first part of Lemma 3 and by Corollary 4 it follows that $DB_{n,1}$ is an anticode of diameter 3 and size $2(n-1)$.

Let $\mathcal{A}$ be an optimal anticode of diameter 3 in S_n, where $n \geq 5$, and let

$$\mathcal{A}_e = \{\sigma \in \mathcal{A} \ : \ w_K(\sigma) \equiv 0 \pmod 2\},$$
$$\mathcal{A}_o = \{\sigma \in \mathcal{A} \ : \ w_K(\sigma) \equiv 1 \pmod 2\}.$$

Since the Kendall τ-metric is bipartite, it follows that $\mathcal{A}_e$ and $\mathcal{A}_o$ are anticodes of diameter 2. If $n \geq 5$ then by Theorem 7 it follows that $|\mathcal{A}_e| \leq n$ ($|\mathcal{A}_o| \leq n$, respectively) and $|\mathcal{A}_e| = n$ ($|\mathcal{A}_0| = n$, respectively) if and only if $\mathcal{A}_e$ ($\mathcal{A}_0$, respectively) is a ball of radius one. The anticodes $\mathcal{A}_e$ and $\mathcal{A}_o$ cannot be balls of radius one and therefore, $|\mathcal{A}_e| \leq n-1$ and $|\mathcal{A}_o| \leq n-1$. Thus, $|\mathcal{A}| = |\mathcal{A}_e| + |\mathcal{A}_o| \leq 2(n-1)$, for $n \geq 5$. ■

As a consequence of Corollary 3 and the fact that $DB_{n,R}$ is an anticode of diameter $2R+1$ we have the following upper bound on $P(n, 2\delta)$, which generally considerably improves the known upper bounds.

Corollary 5.

$$P(n, 2(R+1)) \leq \frac{n!}{|DB_{n,R}|}.$$

Corollary 6.

$$P(n, 4) \leq \frac{n!}{2(n-1)}.$$

Note, that $P(n, 4) \geq \frac{(n)!}{2(2n-1)}$ [21] and hence the size of the best known code is within a factor of two from the new upper bound.

Note also, that since we proved that $DB_{n,1}$ is an optimal anticode of diameter 3, the upper bound of Corollary 6 is the best bound that can be derived from Corollary 3. An intriguing question is whether $B_{n,R}$ is an optimal anticode of diameter $D = 2R$, where $0 \leq R < \frac{\binom{n}{2}}{2}$ and whether $DB_{n,R}$ is an optimal anticode of diameter $2R+1$, where $0 \leq R < \frac{\binom{n}{2}-1}{2}$. Table I present the sizes of the largest known anticodes of diameter D in S_n, for $4 \leq n \leq 12$ and $2 \leq D \leq \max\left\{\binom{n}{2}, 20\right\}$. For even values of D, the bound is the size of the related ball of radius $\frac{D}{2}$ and was computed by computer. A formula to compute some of these values is given in [25], [30] and also in [21]. Odd values of D were computed using Corollary 4. Related bounds on $P(n, d)$ will be presented in Section V.

For completeness, we will present in the next few results some simple optimal anticodes and the related perfect codes and diameter perfect codes in S_n, which might be considered as trivial. If $D = \binom{n}{2}$ then an optimal anticode of diameter D in S_n is S_n itself. Hence, if $\frac{\binom{n}{2}}{2} \leq R < \binom{n}{2}$ then an optimal anticode with diameter $2R \geq \binom{n}{2}$ is S_n. Since $|B_{n,R}| < n!$, for $\frac{\binom{n}{2}}{2} \leq R < \binom{n}{2}$, it follows that $B_{n,R}$ is not an optimal anticode with diameter $2R$. Similarly, if $\frac{\binom{n}{2}-1}{2} \leq R < \binom{n}{2} - 1$ then $|DB_{n,R}| < n!$ and hence, $DB_{n,R}$ is not an optimal anticode with diameter $2R+1$.

Theorem 9. *$\mathcal{A} \subset S_n$ is an optimal anticode of diameter $\binom{n}{2} - 1$ if and only if $\mathcal{A}$ contains either σ or σ^r, for each $\sigma \in S_n$.*

Proof: If $\mathcal{A}$ is an optimal anticode of diameter $\binom{n}{2} - 1$ then by Lemma 1, for every $\sigma \in S_n$, $\mathcal{A}$ cannot contain both σ and σ^r. On the other hand, if $\pi \neq \sigma^r$ then $d_K(\sigma, \pi) \leq \binom{n}{2} - 1$. Thus, the theorem follows. ■

Corollary 7. *An optimal anticode $\mathcal{A} \subset S_n$ of diameter $\binom{n}{2} - 1$ has size $\frac{n!}{2}$ and can be chosen in $2^{\frac{n!}{2}}$ different ways.*

Corollary 8.

- *For each $\sigma \in S_n$, the set $\{\sigma, \sigma^r\}$ is a D-diameter perfect code, $D = \binom{n}{2} - 1$.*
- *If $2R+1 = \binom{n}{2}$ then $\{\sigma, \sigma^r\}$ is a perfect R-error-correcting code.*

Theorem 10. *If $\frac{2}{3}\binom{n}{2} < d \leq \binom{n}{2}$ then $P(n, d) = 2$.*

Proof: Any code of the form $\{\sigma, \sigma^r\}$ has minimum Kendall τ-distance at least d, and therefore $P(n, d) \geq 2$.

Assume to the contrary that $P(n, d) \geq 3$, i.e. there exists a code $\mathcal{C} \subset S_n$ with minimum Kendall τ-distance d and of size 3. Since the Kendall τ-metric is right invariant, we can assume w.l.o.g. that $\mathcal{C} = \{\varepsilon, \sigma, \pi\}$. We have that $d \leq w_K(\sigma)$ and $d \leq w_K(\pi)$ and $d \leq d_K(\sigma, \pi)$. By Lemma 1 we have that $d_K(\sigma, \varepsilon^r) \leq \binom{n}{2} - d$ and $d_K(\pi, \varepsilon^r) \leq \binom{n}{2} - d$. By the triangle inequality it follows that $d_K(\sigma, \pi) \leq 2\binom{n}{2} - 2d < 2\binom{n}{2} - 2\frac{2}{3}\binom{n}{2} < d$. ■

Corollary 9. *If $2R = \binom{n}{2} - 1$ then $B_{n,R}$ is an optimal anticode of diameter $\binom{n}{2} - 1$.*

Proof: Follows from Lemma 1, Theorem 9, and Corollary 7. ■

Lemma 4. *If $2R+1 = \binom{n}{2} - 1$ then $DB_{n,R}$ is an optimal anticode of diameter $\binom{n}{2} - 1$.*

Proof: Recall that ε and $(1, 2)$ are the centers of $DB_{n,R}$. By Theorem 9 it is sufficient to show that for every $\sigma \in S_n$, either $\sigma \in DB_{n,R}$ or $\sigma^r \in DB_{n,R}$. If $w_K(\sigma) \leq R$ then by Lemma 1 $w_K(\sigma^r) = \binom{n}{2} - w_K(\sigma) > R+1$ and therefore, $\sigma \in DB_{n,R}$ and $\sigma^r \notin DB_{n,R}$. Similarly, if $w_K(\sigma) > R+1$ then $\sigma \notin DB_{n,R}$ and $\sigma^r \in DB_{n,R}$. If $w_K(\sigma) = R+1$ then by Lemma 1 $w_K(\sigma^r) = R+1$. By Lemma 2 and since $w_K((1,2)) = 1$ it follows that either $d_K(\sigma, (1,2)) = R$ or $d_K(\sigma, (1,2)) = R+2$. Similarly, either $d_K(\sigma^r, (1,2)) = R$ or $d_K(\sigma^r, (1,2)) = R+2$. By

n \ D	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
4	4	6	9	12	24	-	-	-	-	-	-	-	-	-	-	-	-	-	-
5	5	8	14	20	29	38	49	60	120	-	-	-	-	-	-	-	-	-	-
6	6	10	20	30	49	68	98	128	169	210	259	308	360	720	-	-	-	-	-
7	7	12	27	42	76	110	174	238	343	448	602	756	961	1,166	1,416	1,666	1,947	2,228	2,520
8	8	14	35	56	111	166	285	404	628	852	1,230	1,608	2,191	2,774	3,606	4,438	5,546	6,654	8,039
9	9	16	44	72	155	238	440	642	1,068	1,494	2,298	3,102	4,489	5,876	8,095	10,314	13,640	16,966	21,671
10	10	18	54	90	209	328	649	970	1,717	2,464	4,015	5,566	8,504	11,442	16,599	21,756	30,239	38,722	51,909
11	11	20	65	110	274	438	923	1,408	2,640	3,872	6,655	9,438	15,159	20,880	31,758	42,636	61,997	81,358	113,906
12	12	22	77	132	351	570	1,274	1,978	3,914	5,850	10,569	15,288	25,728	36,168	57,486	78,804	119,483	160,162	233,389

TABLE I: sizes of the largest known anticodes of diameter D in S_n

Lemma 1 we conclude that either $d_K(\sigma,(1,2)) = R$ or $d_K(\sigma^r,(1,2)) = R$. ■

The next theorem can be easily verified.

Theorem 11. *Any set $\{\sigma,\pi\}$ such that $d_K(\sigma,\pi) = 1$ is an optimal anticode of diameter one. The set of all permutations of even Kendall τ-weight, known as the alternating group, A_n, is a 1-diameter perfect code. Similarly, the set of all permutations of odd Kendall τ-weight, $S_n \setminus A_n$, is an 1-diameter perfect code. These codes are the only 1-diameter perfect codes in S_n.*

V. Constructions of Large Codes and a Table of the Bounds

In this section we present two large codes with minimum Kendall τ-distance 3 in S_5 and S_7. These two codes have large automorphism groups and can be represented only by one or two codewords, respectively. We hope that the method in which we constructed these codes can be applied for other values of n and minimum Kendall τ-distance. In addition, we present a table of the lower and upper bounds on $P(n,d)$ for small values of n. Throughout this section the positions and elements of permutations of length n are taken from the set $\{0,1,2,\ldots,n-1\}$ (instead of the set $[n]$).

By Theorem 3, there is no perfect single-error-correcting code in S_5, using the Kendall τ-distance. However, if we add to the set of adjacent transpositions, which defines the Kendall τ-metric, the transposition $(0,n-1)$, we obtain a new metric in which the code $\mathcal{C}_5$, consists of the following 20 codewords, is a perfect single-error-correcting code in S_5.

$$
\begin{array}{llll}
[0,1,2,3,4], & [0,2,4,1,3], & [0,3,1,4,2], & [0,4,3,2,1] \\
[1,2,3,4,0], & [2,4,1,3,0], & [3,1,4,2,0], & [4,3,2,1,0] \\
[2,3,4,0,1], & [4,1,3,0,2], & [1,4,2,0,3], & [3,2,1,0,4] \\
[3,4,0,1,2], & [1,3,0,2,4], & [4,2,0,3,1], & [2,1,0,4,3] \\
[4,0,1,2,3], & [3,0,2,4,1], & [2,0,3,1,4], & [1,0,4,3,2]
\end{array}
$$

Note, that if $[\sigma(0),\sigma(1),\ldots,\sigma(4)]$ is a codeword then $[\sigma(1),\ldots,\sigma(4),\sigma(0)]$ and $[2\sigma(0),2\sigma(1),\ldots,2\sigma(4)]$ are also codewords, where the computations are performed modulo 5. Hence, this code can be represented by only one codeword $[0,1,2,3,4]$ and it has an automorphism group of size 20. Note, also that the minimum Kendall τ-distance of this code is at least 3 (since the Kendall τ-distance can only be increased by removing the transposition (0,n-1)) and hence ,

Theorem 12.

$$P(5,3) \geq 20.$$

In general, we suggest to search for codes in S_n, for small n, n prime, and small minimum Kendall τ-distance as follows. We require that if $\sigma = [\sigma(0),\sigma(1),\ldots,\sigma(n-1)]$ is a codeword in the code $\mathcal{C}$ then $[\sigma(1),\ldots,\sigma(n-1),\sigma(0)]$, $[\sigma(0)-1,\sigma(1)-1,\ldots,\sigma(n-1)-1]$, and $[\alpha\sigma(0),\alpha\sigma(1),\ldots,\alpha\sigma(n-1)]$ are also codewords, where the computations are done modulo n and α is a primitive root modulo n. Note, that $[\sigma(0)-1,\sigma(1)-1,\ldots,\sigma(n-1)-1] = \sigma\circ[1,2,\ldots,n-1,0]$. A computer search for such a code is easier since the code has a large automorphism group. We leave as a nice exercise to the reader to verify that a codeword in such a code represents either $n(n-1)$ codewords (if and only if $[0,1,\ldots,n-1]$ is one of the represented codewords, as in $\mathcal{C}_5$) or $n^2(n-1)$ codewords.

Theorem 13.

$$P(7,3) \geq 588.$$

Proof: Verify that the two representatives $\mu = [0,1,2,4,3,6,5]$ and $\nu = [0,1,2,3,6,4,5]$ yield the require code of size 588. ■

The previous known lower bounds on $P(5,3)$ and $P(7,3)$ were 18 and 526, respectively [21]. We summarise with the best known bounds on $P(n,d)$, for $5 \leq n \leq 7$ and $3 \leq d \leq 9$, which are presented in Table II.

VI. Conclusions and Open Problems

We have considered several questions related to bounds on the size of codes in the Kendall τ-metric. We gave a novel technique to exclude the existence of perfect single-error-correcting codes using the Kendall τ-metric. We applied this technique to prove that there are no perfect single-error-correcting codes in S_n, where $n > 4$ is a prime or $4 \leq n \leq 10$, using the Kendall τ-metric. We examine the existence question of diameter perfect codes in S_n and the sizes of optimal anticodes with the Kendall τ-distance. We obtained a new upper bound on the size of a code in S_n with even Kendall

n \ d	3	4	5	6	7	8	9
5	$^{f}20-23^{b}$	$^{h}10-15^{c}$	$^{d}6-8^{a}$	$^{j}4-6^{c}$	$^{i}2^{i}$	$^{i}2^{i}$	$^{i}2^{i}$
6	$^{d}90-119^{b}$	$^{h}45-72^{c}$	$^{d}23-36^{a}$	$^{h}12-24^{c}$	$^{d}10-14^{a}$	$^{h}5-10^{c}$	$^{d}4-7^{a}$
7	$^{e}588-719^{b}$	$^{h}294-420^{c}$	$^{d}110-186^{a}$	$^{h}55-120^{c}$	$^{d}34-66^{a}$	$^{h}17-45^{c}$	$^{d}14-28^{a}$

- a - The sphere packing bound.
- b - The sphere packing bound + Theorem 3.
- c - Corollary 5.
- d - Lower bounds from [21].
- f - Theorem 12.
- e - Theorem 13.
- h - $P(n, 2\delta) \geq \frac{1}{2}P(n, 2\delta-1)$ [21].
- i - Theorem 10.
- j - $\mathcal{C} = \{[1,2,3,4,5],[1,5,2,3,4],[2,3,4,1,5],[1,4,3,2,5]\}$.

TABLE II: Best known lower and upper bound on $P(n,d)$.

τ-distance. Finally, we constructed two large codes with large automorphism groups in S_5 and S_7.

Our discussion raises many open problems from which we choose a few as follows.

1) Prove the nonexistence of perfect codes in S_n, using the Kendall τ-metric, for more values of n and/or other distances.
2) Do there exist more D-diameter perfect codes in S_n with the Kendall τ-metric, for $2 \leq D < \binom{n}{2} - 1$? We conjecture that the answer is no.
3) Is a ball with radius R in S_n always optimal as an anticode with diameter $2R$ in S_n, for $2 \leq R < \frac{\binom{n}{2}}{2}$?
4) Is the double ball with radius R in S_n always optimal as an anticode with diameter $2R+1$ in S_n, for $2 \leq R < \frac{\binom{n}{2}-1}{2}$?
5) What is the size of an optimal anticode in S_n with diameter D?
6) Improve the lower bounds on the sizes of codes in S_n with even minimum Kendall τ-distance.
7) Can the codes in S_5 and S_7 from Section V be generalized for higher values of n and to larger distances? Are these codes of optimal size?

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APPENDIX A

In Theorem 3 we proved that a perfect single-error-correcting code in S_n with the Kendall τ-metric does not exist if $n > 4$ is a prime or if $n = 4$. The proof of Theorem 3 is based on a certain linear equations system, where the existence of a perfect single-error-correcting code in S_n implies the existence of a solution to the linear equations system over the integers, and thus, by showing the nonexistence of such solution we derive the nonexistence of a perfect single-error-correcting code. By using similar techniques we prove the nonexistence of perfect single-error-correcting codes in S_n for $n \in \{6, 8, 9, 10\}$. For each such n, let $\mathcal{C}$ be a perfect single-error-correcting code in S_n. We will describe the corresponding linear equations system and use a computer to show that this linear equations system does not have a solution over the integers.

1) $n = 6$**:** We denote by D_6 the set of all vectors of $\{1,2,3\}^6$ in which each of the elements 1,2,3 appears twice. For each $\mathbf{v} \in D_6$ we define $S_{\mathbf{v}}$ to be the set of eight permutations in S_6, such that the elements 1 and 2 appear in the two positions in which 1 appears in $\mathbf{v}$, the elements 3 and 4 appear in the two positions in which 2 appears in $\mathbf{v}$, and the elements 5 and 6 appear in the two positions in which 3 appears in $\mathbf{v}$. Let $x_{\mathbf{v}} = |\mathcal{C} \cap S_{\mathbf{v}}|$ and let $\mathbf{x} = (x_{\mathbf{v}_1}, x_{\mathbf{v}_2}, \ldots, x_{\mathbf{v}_m})$, where $m = |D_6| = \frac{6!}{2!2!2!}$. By considering how the elements of $S_{\mathbf{v}}$ are covered (similarly to the way it was done in the proof of Theorem 3), for each $\mathbf{v} \in D_6$, we obtain a linear equations system of the form $A\mathbf{x}^T = |S_{\mathbf{v}}| \cdot \mathbf{1} = 8 \cdot \mathbf{1}$, where A is a square matrix of order m. The kernel of A is an one-dimensional vector space which is spanned by a vector $\mathbf{y} \in \{0,-1,1\}^9$, that has both negative and positive entries. Every solution for this system is of the form $\frac{8}{6} \cdot \mathbf{1} + \alpha \cdot \mathbf{y}$, $\alpha \in \mathbb{R}$, and therefore, the system does not have a solution in which all entries are integers.

2) $n = 8$**:** We denote by D_8 the set of all vectors $\mathbf{v} \in \{1,2,3,4\}^8$ in which each of the elements 1 and 2 appears three times and each of the elements 3 and 4 appears once. For every $\mathbf{v} \in D_8$ we define $S_{\mathbf{v}}$ to be the set of 36 permutations in S_8, such that the elements $1, 2,$ and 3 appear in the three positions in which 1

appears in $\mathbf{v}$, the elements $4, 5$, and 6 appear in the three positions in which 2 appears in $\mathbf{v}$, the element 7 appears in the position of 3 in $\mathbf{v}$, and the element 8 appears in the position of 4 in $\mathbf{v}$. Let $x_{\mathbf{v}} = |\mathcal{C} \cap S_{\mathbf{v}}|$ and let $\mathbf{x} = (x_{\mathbf{v}_1}, x_{\mathbf{v}_2}, \ldots, x_{\mathbf{v}_m})$, where $m = |D_8| = \frac{8!}{3!3!}$. By considering how elements of $S_{\mathbf{v}}$ are covered, for each $\mathbf{v} \in D_8$, we obtain a linear equations system of the form $A\mathbf{x}^T = 36 \cdot \mathbf{1}$, where A is a square matrix of order m. The system has a unique solution, $\mathbf{x}^T = \frac{36}{8} \cdot \mathbf{1}$, which has non-integer entries.

3) $n = 9$**:** We denote by D_9 the set of all vectors $\mathbf{v} \in \{1, 2, 3\}^9$ in which the element 1 appears five times and each of the elements 2 and 3 appears twice. For every $\mathbf{v} \in D_9$ we define $S_{\mathbf{v}}$ to be the set of 480 permutations in S_8, such that the elements $1, 2, 3, 4,$ and 5 appear in the five positions in which 1 appears in $\mathbf{v}$, the elements 6 and 7 appear in the two positions in which 2 appears in $\mathbf{v}$, and the elements 8 and 9 appear in the two positions in which 3 appears in $\mathbf{v}$. Let $x_{\mathbf{v}} = |\mathcal{C} \cap S_{\mathbf{v}}|$ and let $\mathbf{x} = (x_{\mathbf{v}_1}, x_{\mathbf{v}_2}, \ldots, x_{\mathbf{v}_m})$, where $m = |D_9| = \frac{9!}{5!2!2!}$. By considering how elements of $S_{\mathbf{v}}$ are covered, for each $\mathbf{v} \in D_9$, we obtain a linear equations system of the form $A\mathbf{x}^T = 480 \cdot \mathbf{1}$, where A is a square matrix of order m. The system has a unique solution, $\mathbf{x}^T = \frac{480}{9} \cdot \mathbf{1}$, which has non-integer entries.

4) $n = 10$**:** We denote by D_{10} the set of all vectors $\mathbf{v} \in \{1, 2, 3\}^{10}$ in which each of the elements 1 and 2 appears four times and the element 3 appears twice. For every $\mathbf{v} \in D_{10}$ we define $S_{\mathbf{v}}$ to be the set of 1,152 permutations in S_{10}, such that the elements $1, 2, 3,$ and 4 appear in the four positions in which 1 appears in $\mathbf{v}$, the elements $5, 6, 7,$ and 8 appear in the four positions in which 2 appears in $\mathbf{v}$, and the elements 9 and 10 appear in the two positions in which 3 appears in $\mathbf{v}$. Let $x_{\mathbf{v}} = |\mathcal{C} \cap S_{\mathbf{v}}|$ and let $\mathbf{x} = (x_{\mathbf{v}_1}, x_{\mathbf{v}_2}, \ldots, x_{\mathbf{v}_m})$, where $m = |D_{10}| = \frac{10!}{4!4!2!}$. By considering how elements of $S_{\mathbf{v}}$ are covered, for each $\mathbf{v} \in D_{10}$, we obtain a linear equations system of the form $A\mathbf{x}^T = 1,152 \cdot \mathbf{1}$, where A is a square matrix of order m. The system has a unique solution, $\mathbf{x}^T = \frac{1,152}{10} \cdot \mathbf{1}$, which has non-integer entries.

APPENDIX B

The purpose of this appendix is to prove Theorem 7 given in Section IV.

Theorem 7. *Every optimal anticode with diameter 2 (using the Kendall τ-distance) in S_n, $n \geq 5$, is a ball with radius one whose size is n.*

Lemma 5. *Let $\sigma = (i, i+1) \circ (i+1, i+2)$ and let $\rho \neq \sigma$ be a permutation of weight 2 and distance 2 from σ. Then $\rho = (j, j+1) \circ (i+1, i+2)$ or $\rho = (i+1, i+2) \circ (i, i+1)$.*

Proof: Recall first that for any two permutations α, β, $d_K(\alpha, \beta) = 1$ if and only if there exists an adjacent transposition $(k, k+1)$, such that $\alpha = (k, k+1) \circ \beta$. We distinguish between four cases. In the first two cases the permutation ρ is at distance 2 from σ.

I. $\rho = (j, j+1) \circ (i+1, i+2)$. In this case $\sigma = (i, i+1) \circ (j, j+1) \circ \rho$ and therefore $d_K(\sigma, \rho) \leq 2$. By Lemma 2 we have that the Kendall τ-metric is bipartite and since σ and ρ are both of even weight it follows that $d_K(\sigma, \rho) \geq 2$. Thus, $d_K(\sigma, \pi) = 2$.

II. $\rho = (i+1, i+2) \circ (i, i+1)$. In this case we have that $\sigma = \rho \circ \rho$ and similarly it follows that $d_K(\sigma, \rho) = 2$.

III. If $\rho = (j, j+1) \circ (k, k+1)$, where $j \neq k$ and $j, k \neq i+1$, then by (1) we have that $d_K(\sigma, \rho) \geq |\{(i+2, i), (i+2, i+1), (k, k+1)\}| > 2$.

IV. If $\rho = (i+1, i+2) \circ (j, j+1)$. We distinguish be between four subcases.
   1) If $j \notin \{i, i+1, i+2\}$, then $\rho = (j, j+1) \circ (i+1, i+2)$ and this case was considered in I.
   2) $j = i$ was considered in II.
   3) If $j = i+1$ then $\rho = \varepsilon$, i.e $w_K(\rho) = 0$.
   4) If $j = i+2$ then $\rho = (i+1, i+2) \circ (i+2, i+3)$ and by (1) we have $d_K(\sigma, \rho) = |\{(i+2, i), (i+2, i+1), (i+1, i+3), (i+2, i+3)\}| = 4$.

■

Lemma 6. *Let $\sigma = (i, i+1) \circ (i+1, i+2)$ and $\pi = (i+1, i+2) \circ (i, i+1)$, where $i \in [n-2]$, and let ρ be a permutation of weight 2, $\rho \neq \sigma$ and $\rho \neq \pi$. Then either $d_K(\sigma, \rho) \geq 4$ or $d_K(\pi, \rho) \geq 4$.*

Proof: By Lemma 5 it follows that if $d_K(\sigma, \rho) = 2$ then $\rho = (j, j+1) \circ (i+1, i+2)$ or $\rho = \pi$. By symmetry it follows that if $d_K(\pi, \rho) = 2$ then $\rho = (j, j+1) \circ (i, i+1)$ or $\rho = \pi$. Hence, there is no permutation ρ of weight 2 and distance 2 from both σ and π. By Lemma 2 we also have that the Kendall τ-metric is bipartite and we conclude that any permutation of weight 2 other then σ and π must be at distance at least four from σ or π. ■

Lemma 7. *Let $\mathcal{A}$ be an anticode in S_n with diameter 2 such that $\varepsilon \in \mathcal{A}$, and let $\mathcal{B}$ be the set of all permutations of weight 2 in $\mathcal{A}$. If $|\mathcal{B}| \geq 4$ then $\mathcal{B}$ is contained in a ball of radius one centered at some permutation $\sigma \in S_n$ of weight one.*

Proof: If there exists some $i \in [n-2]$ such that $(i, i+1) \circ (i+1, i+2), (i+1, i+2) \circ (i, i+1) \in \mathcal{B}$, then by Lemma 6 any other permutation of weight 2 is at distance at least four from either $(i, i+1) \circ (i+1, i+2)$ or $(i+1, i+2) \circ (i, i+1)$, and therefore $|\mathcal{B}| = 2$.

If for some $i \in [n-2]$ either $(i, i+1) \circ (i+1, i+2)$ or $(i+1, i+2) \circ (i, i+1)$ belongs to $\mathcal{B}$, say w.l.o.g. $(i, i+1) \circ (i+1, i+2) \in \mathcal{B}$, then every permutation of $\mathcal{B} \setminus \{(i, i+1) \circ (i+1, i+2)\}$ must be at distance 2 from $(i, i+1) \circ (i+1, i+2)$, and by Lemma 5 it follows that every such permutation must be of the form $(j, j+1) \circ (i+1, i+2)$ for some $j \notin \{i, i+1\}$. Therefore, $\mathcal{B} \subset B((i+1, i+2), 1)$.

If each permutation of $\mathcal{B}$ is a multiplication of two disjoint adjacent transpositions then let $\rho = (i, i+1) \circ (j, j+1) \in \mathcal{B}$, where $j \notin \{i-1, i, i+1\}$. Hence, all permutations of $\mathcal{B}$ are of the form $(\ell, \ell+1) \circ (j, j+1)$, where $\ell \notin \{j, j+1\}$, or $(\ell, \ell+1) \circ (i, i+1)$, where $\ell \notin \{i, i+1\}$. Assume w.l.o.g. that $\pi = (\ell, \ell+1) \circ (j, j+1) \in \mathcal{B}$, $\pi \neq \rho$. If every permutation of $\mathcal{B}$ is of the form $(k, k+1) \circ (j, j+1)$ then $\mathcal{B} \subset B((j, j+1), 1)$. Otherwise, the only possible other permutation of $\mathcal{B}$ is $(i, i+1) \circ (\ell, \ell+1)$ and hence $|\mathcal{B}| \leq 3$.

Thus, if $|\mathcal{B}| \geq 4$ then $\mathcal{B} \subset B(\sigma, 1)$, for some σ of weight one. ■

Proof of Theorem 7: Let $\mathcal{A} \subset S_n$, $n \geq 5$, be an anticode of diameter 2. The Kendall τ-metric is right invariant and hence w.l.o.g. we can assume that $\varepsilon \in \mathcal{A}$. Therefore, all the permutations of $\mathcal{A}$ are of weight at most two. We distinguish between four cases:

Case 1: If $\mathcal{A}$ does not contain a permutation of weight one then by Lemma 7 it follows that $\mathcal{A}$ is contained in a ball of radius one centered at a permutation of weight one or $|\mathcal{A}| \leq 4$.

Case 2: If $\mathcal{A}$ contains exactly one permutation $\sigma \in S_n$ of weight one then by Lemma 2, the distance between σ and any permutation of weight 2 is an odd integer and therefore, all permutations of weight 2 in $\mathcal{A}$ must be at distance one from σ. Thus, $\mathcal{A} \subseteq B(\sigma, 1)$.

Case 3: If $\mathcal{A}$ contains two permutations of weight one, $\sigma = (i, i+1)$ and $\pi = (j, j+1)$, where σ and π are disjoint transpositions, then the only permutation of weight 2 and distance one from both σ and π is $(i, i+1) \circ (j, j+1)$ and therefore $\mathcal{A}$ cannot contain more than one permutation of weight 2, hence $|\mathcal{A}| \leq 4$.

Case 4: If $\mathcal{A}$ contains two permutations of weight one, $\sigma = (i, i+1)$ and $\pi = (i+1, i+2)$, for some $i \in [n-2]$, then there is no permutation of weight 2 and distance one from both σ and π and therefore $\mathcal{A}$ cannot contain permutations of weight 2, hence $|\mathcal{A}| \leq 3$.

Case 5: If $\mathcal{A}$ contains at least three permutations of weight one then $\mathcal{A}$ cannot contain permutations of weight 2 and therefore $\mathcal{A} \subseteq B(\varepsilon, 1)$.

Thus, we proved that either $\mathcal{A}$ is contained in a ball of radius one or $|\mathcal{A}| \leq 4$. Since the size of a ball of radius one in S_n is n, it follows that if $n \geq 5$ then every optimal anticode of diameter 2 in S_n is a ball of radius one. □

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