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Integer Partitions and Binary Trees

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IN MEMORY OF RODICA SIMION

We present observations and problems connected with a weighted binary tree representation of integer partitions.

Key Words: integer partition; 2-core; 2-quotient; Ferrers diagram; binary tree.

1. INTRODUCTION

The central theme in Rodica Simion's work in recent years was noncrossing partitions, in which the Catalan numbers feature prominently. Rodica's enduring interest in the Catalan family was reinforced by the proximity of fellow Catalan devotee Lou Shapiro; the combined breadth of their knowledge of the Catalan family was virtually exhaustive.

Another long-term interest, the symmetric group, was sparked early on by reading the following intriguing problem in the short and remarkably thought-provoking paper of Rothaus and Thompson [7]. Indeed, she often referred to this as *the* S_n problem.

Let G_n *be the Cayley graph of* S_n *using transpositions as generators. Can the vertices of* G_n *be partitioned into* 1-*spheres?*

Rothaus and Thompson give a partial solution. (See also [1].) This problem remains open, as is true of a number of variations that have been investigated.

This note poses problems in an area that Rodica was in a preliminary stage of developing that touch on the topics mentioned above. The link with noncrossing partitions and the Catalan numbers arises through binary trees; the connection with the symmetric group comes about via integer partitions and their cores and quotients.

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We hope that by presenting this material, others may bring to fruition the ideas she was developing. We realize that we are publicizing mere sketches that, had she had the opportunity to pursue them, would have been significantly developed and highly polished before publication. Thus, if certain directions for research mentioned in this paper prove to be false leads, this should be taken solely as a reflection of the preliminary state of the area and the author's possibly faulty reconstruction of conversations with Rodica.

These problems concern a representation of integer partitions by weighted binary trees. The necessary background is given in Section 2 and the problems are stated in Section 3. Throughout this paper we wish to emphasize the utility of weighted binary tree representations of partitions rather than the novelty of the results.

2. BACKGROUND

We review the notions of the 2-core and 2-quotient of an integer partition. A full account of these topics can be found in [3, pp. 75–85].

Let λ be an integer partition of n, written $|\lambda| = n$. The 2-core of λ is formed as follows. Draw the Ferrers diagram of λ and successively remove 2-hooks ($\square\square$ or $\substack{\square \\ \square}$), at each stage leaving a valid Ferrers diagram, until no more 2-hooks can be removed. The Ferrers diagram that remains is independent of the order in which the 2-hooks are removed. This remaining diagram, which may be empty, is the 2-*core* of λ. (See Fig. 1.) The 2-core of λ necessarily has triangular shape. Thus, the weight of the 2-core is a triangular number, $k(k+1)/2$, for some k with $k \geq 0$. Observe that a partition and its conjugate have the same 2-core.

The 2-quotient of a partition λ is an ordered pair of partitions (β_0, β_1) formed as follows. In the diagram for λ, write 0s and 1s in a checkerboard fashion, starting with a 0 in the top, left-most box. Draw a horizontal line through any row that ends in 0 and a vertical line through any column that ends in 1. The diagram of β_0 consists of the boxes that are contained in

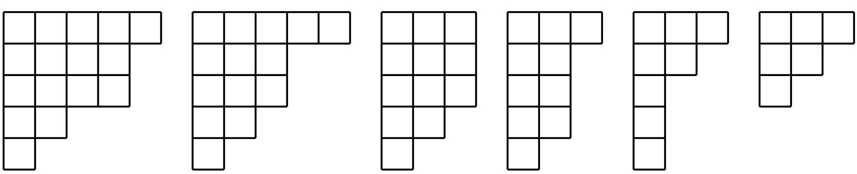

FIG. 1. One of the several possible sequences of deletions of 2-hooks that yield the 2-core of (5, 4, 4, 2, 1).

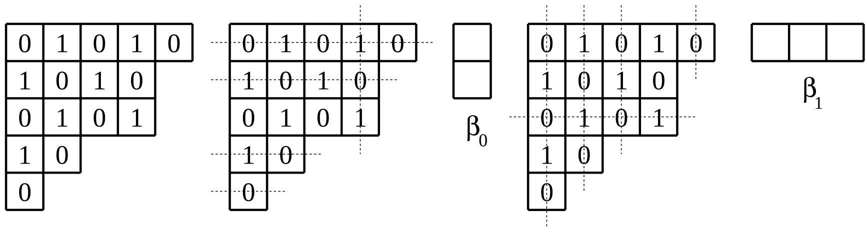

FIG. 2. The 2-quotient of (5, 4, 4, 2, 1).

both a vertical and a horizontal line; these boxes are then pushed toward the northwest, so as to be justified with respect to the left and top, to form a Ferrers diagram. To form β_1, carry out the same procedure with the roles of rows and columns reversed (i.e., lines are drawn through rows that end in 1 and columns that end in 0). (See Fig. 2.) It is immediate that if the 2-quotient of λ is (β_0, β_1), then the 2-quotient of its conjugate λ' is (β_1', β_0').

We write the 2-core α and 2-quotient (β_0, β_1) of λ as a triple $(\alpha; \beta_0, \beta_1)$. A key relation is

$$|\lambda| = |\alpha| + 2|\beta_0| + 2|\beta_1|. \tag{1}$$

In particular, both β_0 and β_1 have strictly smaller weight than λ. Note that $|\lambda|$ imposes restrictions on the triangular number $|\alpha|$ that can form the 2-core since $|\lambda| \equiv |\alpha| (\mathrm{mod}\, 2)$.

A basic result [6] (see also [3, p. 83, Theorem 2.7.30]) is that the triple $(\alpha; \beta_0, \beta_1)$ uniquely determines λ; this justifies writing $\lambda = (\alpha; \beta_0, \beta_1)$. By the remarks above, a partition λ and its conjugate λ' are related as follows:

$$\lambda = (\alpha; \beta_0, \beta_1) \qquad \text{if and only if} \qquad \lambda' = (\alpha; \beta_1', \beta_0'). \tag{2}$$

Construct a binary tree for $\lambda = (\alpha; \beta_0, \beta_1)$ as follows. First, let the root be α, let the left child be β_0, and let the right child be β_1 (any of which could be $\emptyset$). Next, iterate the construction starting from each of the leaves (initially β_0 and β_1). Since $|\beta|$ strictly decreases, this process terminates. Finally, delete any empty leaves and label each vertex with its weight (a triangular number). This gives a unique weighted binary tree representation for λ. (See Fig. 3.)

From Eq. (1), we get

$$|\lambda| = \sum_v 2^{\mathrm{depth}\,(v)} \cdot \mathrm{weight}(v), \tag{3}$$

where the sum is taken over all vertices v in the tree. The construction above gives a one-to-one correspondence between integer partitions and binary trees whose vertices are weighted with triangular numbers, including 0, with the restriction that leaves must have non-zero weight.

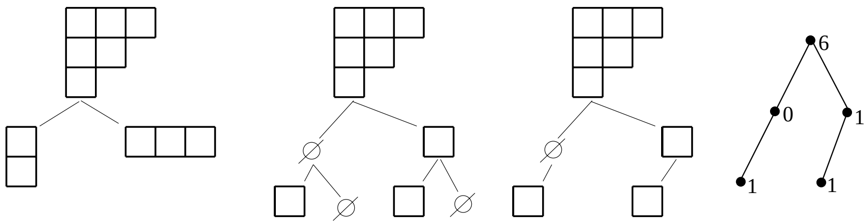

FIG. 3. Construction of the binary tree representation for $(5, 4, 4, 2, 1)$.

EXAMPLE 2.1. For the partition $\lambda = ((2^n)^{2^n})$ of 4^n in which each of the 2^n parts is 2^n, the corresponding tree is a complete binary tree of depth n in which all leaves have depth n and are labeled 1, and all other vertices are labeled 0. (See Fig. 4(a).)

EXAMPLE 2.2. For the partition $\lambda = (n)$ of n into a single part, the corresponding tree is a twisted version of the binary representation of n. Specifically, for each 0 in the binary representation of n, there is a vertex labeled 0 that has no left child, and for each 1 in the binary representation of n, there is a vertex labeled 1 that has no right child. (See Fig. 4(b).)

EXAMPLE 2.3. Consider the partition $\lambda = (n^2)$ of $2n$ into two equal parts. The 2-core is 0. If n is even, then $\beta_0 = \beta_1 = (n/2)$. If n is odd, then $\beta_0 = ((n+1)/2)$ and $\beta_1 = ((n-1)/2)$. In either case, Example 2.2 tells how to find the subtrees for β_0 and β_1. This example can be generalized in a variety of ways.

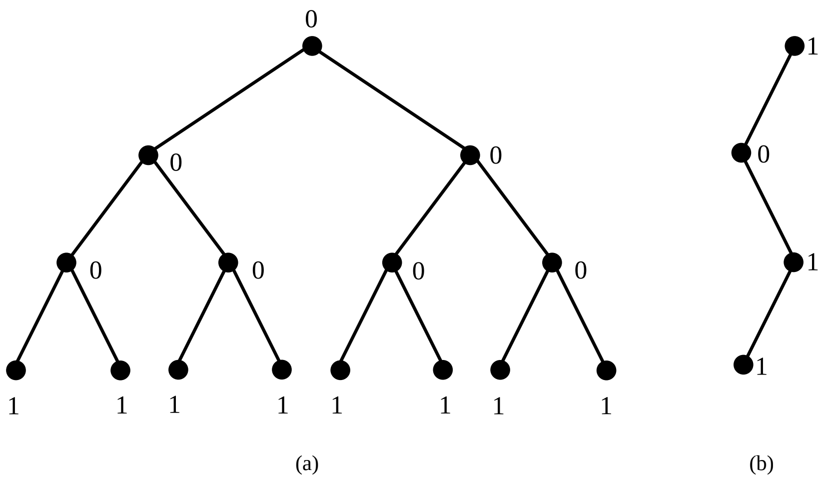

FIG. 4. The binary tree representation for (a) the partition $((8)^8)$ of 64 and (b) the partition (13) of $13 = (1101)_2$.

3. PROBLEMS

Let $g(x) = x + x^3 + x^6 + x^{10} + \cdots$ be the generating function for the non-zero triangular numbers, and let $G(x) = 1 + g(x)$ be the generating function for all triangular numbers. A well-known identity, due to Gauss, is

$$G(x) = \frac{\prod_{n\geq 1}(1 - x^{2n})}{\prod_{n\geq 1}(1 - x^{2n-1})}.$$

This also follows from the unique decomposition of λ into the triple $(\alpha; \beta_0, \beta_1)$ given above.

From Eq. (3), we get the following generating function identity that connects integer partitions and binary trees:

$$\sum_{n\geq 0} p(n)x^n = \sum_{\substack{\text{binary} \\ \text{trees } T}} \left(\prod_{\substack{\text{internal} \\ \text{vertices} \\ v \text{ of } T}} G\left(x^{2^{\text{depth}(v)}}\right) \cdot \prod_{\substack{\text{leaves} \\ v \text{ of } T}} g\left(x^{2^{\text{depth}(v)}}\right)\right). \tag{4}$$

Here, as usual, $p(n)$ is the number of partitions of n. Also, the sum on the right includes the empty binary tree, which contributes 1. To clarify the product, we note that the unlabeled counterpart of the rightmost binary tree in Fig. 3 contributes $G(x) \cdot (G(x^2))^2 \cdot (g(x^4))^2$ to this sum.

That the Catalan numbers enumerate binary trees motivates the following question.

Problem 1. Can Eq. (4) be recast in a useful way as some type of Catalan generating function? Of course, the left side of Eq. (4) is not an algebraic function, so this limits what can be done. Hence one might instead consider restricted classes of partitions defined by limiting the binary trees in the sum, the weights in the product, or both.

One obtains many weight-preserving involutions on the set of all partitions as follows. Pick a vertex v of the infinite complete binary tree and map λ to λ^* if the binary tree representations of λ and λ^* are related by switching the left and right subtrees (if any) of v. This operation preserves the depth of each vertex, and so, by Eq. (3), fixes the weight; i.e., $|\lambda| = |\lambda^*|$. Each choice of a vertex v gives rise to such an involution, and these involutions generate a group of weight-preserving bijections on the set of all partitions.

Problem 2. How many orbits of partitions are there, as a function of n, under this group action?

Note that the size of each orbit is a power of 2 (possibly 2^0). To make this observation more precise, call a vertex of a weighted binary tree *active* if its left and right subtrees, when viewed as weighted binary trees in their own right, are in different orbits. Let $a(T)$ denote the number of active vertices of a weighted binary tree T. An easy induction on depth gives the following result.

Observation 1. The size of the orbit of a weighted binary tree T is $2^{a(T)}$.

It follows that the singleton orbits correspond precisely to completely symmetric binary trees, by which we mean complete binary trees in which all leaves have the same depth and all vertices of the same depth have the same weight. (Figure 4(a) shows a completely symmetric binary tree.) It follows that the number $p(n)$ of partitions of n is congruent modulo 2 to the number of completely symmetric binary trees of total weight n. In the case of completely symmetric binary trees, Eq. (3) becomes

$$|\lambda| = \sum_{d\geq 0} 2^d \cdot 2^d \cdot \text{weight}(d) = \sum_{d\geq 0} 4^d \cdot \text{weight}(d), \tag{5}$$

where the sums are over depths, d, and we collect all 2^d vertices of depth d into a single term. It follows that the generating function for completely symmetric binary trees is $G(x) \cdot G(x^4) \cdot G(x^{16}) \cdots$. Thus, we have the congruence

$$\sum_{n\geq 0} p(n)x^n \equiv \prod_{k\geq 0} G(x^{4^k}) \qquad (\text{mod}\, 2). \tag{6}$$

Although there are other ways of obtaining congruence (6), using weighted binary trees gives a nice visual derivation.

In passing, we remark that determining the overall distribution of the parity of $p(n)$ is considered to be a very difficult problem.

In [8], it is shown that the sequence $\frac{1}{2}(p(n) - q_0(n))$, where $q_0(n)$ is the number of self-conjugate partitions, is even infinitely often and odd infinitely often. The current setting suggests the following related problem.

Problem 3. Let $r(n)$ be the number of completely symmetric binary trees of weight n. Is the sequence $\frac{1}{2}(p(n) - r(n))$ even infinitely often and odd infinitely often?

One can also investigate the value of $p(n)$ modulo higher powers of 2. We give one example. Consider $p(n)$ modulo 4. Since $p(n)$ is the sum of the sizes of the orbits of the binary trees that represent n and since the size of each orbit is a power of 2, it follows that to determine $p(n)$ modulo 4, it would be sufficient to know the orbits that have one or two elements. By Observation 1, this amounts to enumerating the weighted binary trees that have at most one active vertex. Above we noted that the binary trees that have no active vertices are the completely symmetric binary trees. Note that if a vertex is active, then at least one of its parents and its sibling (if such exists) is active. It follows that if a binary tree has precisely one active vertex, then that vertex must be the root. Therefore we get the following congruence, where the terms on the right correspond, respectively, to the root and its children:

$$\sum_{n\geq 0} p(n)x^n \equiv G(x) \prod_{k\geq 0} \left(G(x^{2\cdot 4^k})\right)^2 \qquad (\mathrm{mod}\, 4).$$

From Eq. (2), it follows that the operation of conjugation corresponds to interchanging left and right throughout the binary tree, i.e., the binary trees associated with λ and λ' are mirror images. For instance, from the binary tree representation of $\lambda = (n)$ noted in Example 2.2, we obtain the binary tree representation of $\lambda' = (1^n)$ using the following rule: for each 0 in the binary representation of n, there is a vertex labeled 0 that has no right child, and for each 1 in the binary representation of n, there is a vertex labeled 1 that has no left child.

It follows that a completely symmetric binary tree necessarily corresponds to a self-conjugate partition. However, the converse is false. For example, it follows from Eq. (5) that there are no completely symmetric binary trees of weight 8; however, there are two self-conjugate partitions of 8, namely (4, 2, 1, 1) and (3, 3, 2).

The next question is motivated by the observation above that conjugation corresponds to the associated pair of binary trees being mirror images.

Problem 4. Which symmetries of trees have nice descriptions in terms of Ferrers diagrams?

The next problem concerns statistics such as the depth of trees. In connection with the depth, observe that Eq. (1) implies that the depth of the binary tree representation of any partition of n is at most $1 + \log_2(n)$.

Problem 5. Determine the distribution of various statistics associated with the binary tree representation of partitions. For instance, for each n, consider the binary tree representation for partitions of n and determine the distribution of the numbers of vertices (the sizes of the trees) or the depths of these trees.

Many problems have been posed, and many solved, about parameters such as the number of parts and the size of the parts in the case of linear representations of partitions, $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. (See, e.g., [5, Vol. 4, Section P68].) One can consider similar questions for parameters such as the number of vertices and the weights for binary tree representations of partitions. It remains to be seen which of these questions are sufficiently natural to permit reasonable answers.

With a partition λ of n, we get another partition $\psi(\lambda)$ of n as follows. First, construct the weighted binary tree representation of λ. For each vertex v of the binary tree, we have the part $2^{\mathrm{depth}(v)} \cdot \mathrm{weight}(v)$ of $\psi(\lambda)$. For instance, for the partition $\lambda = (5, 4, 4, 2, 1)$ of 16 shown in Fig. 3, we get $\psi(\lambda) = (6, 4, 4, 2)$.

Consider the functional digraph Γ_n define by ψ as follows. The vertices are the partitions of n; the directed edges are the ordered pairs $(\lambda, \psi(\lambda))$. The following questions suggest some of the many structural properties one might investigate for Γ_n.

Problem 6. How many components does Γ_n have? What are the fixed points of ψ? Which fixed points of ψ are isolated vertices of Γ_n? Is there a non-trivial bound on the lengths of cycles in Γ_n?

We note that the partitions in Example 2.1 form an infinite family of fixed points of ψ. One can also check that these are isolated vertices of Γ_n.

Note that if λ_1 and λ_2 are in the same orbit under the group action considered in Problem 2, then $\psi(\lambda_1) = \psi(\lambda_2)$. Thus, the number of components in Γ_n is at most the number of orbits. Indeed, small examples suggest that the number of components in Γ_n is significantly less than the number of orbits. For instance, for $n = 7$, there are five orbits but only two components. One might ask if the ratio of the number of components to the number of orbits tends to zero as n becomes large.

We note that questions similar to those posed in this paper can be asked if one starts instead with p-cores and p-quotients. In contrast to 2-cores, where there is a (unique) 2-core of weight n if and only if n is a triangular number, if $p > 2$ there can be many p-cores of a given weight n. The generating function for the number of p-cores of a given weight is known to be

$$G_p(x) = \frac{\prod_{n\geq 1}(1 - x^{pn})^p}{\prod_{n\geq 1}(1 - x^n)}. \tag{7}$$

By choosing p to be a prime and defining a group action on labeled p-ary trees by cyclically permuting subtrees, we obtain the congruence

$$\sum_{n\geq 0} p(n)x^n \equiv \prod_{k\geq 0} G_p(x^{p^{2k}}) \pmod{p},$$

of which Eq. (6) is a special case.

Since congruences concerning $p(n)$ bring to mind the classical results of Ramanujan, we end with a brief comment in this direction (for a thorough treatment, see [2]). Specifically, recall that Ramanujan proved the congruence

$$p(5m + 4) \equiv 0 \pmod{5}. \tag{8}$$

To develop a tree interpretation of this, we will consider 5-ary trees. For a partition λ of a number of the form $5m + 4$, consider its weighted 5-ary tree representation. In place of Eq. (1), we have

$$5m + 4 = |\lambda| = |\alpha| + 5(|\beta_0| + |\beta_1| + |\beta_2| + |\beta_3| + |\beta_4|),$$

where α is the 5-core of λ and $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$ is the 5-quotient of λ. We therefore get the congruence $|\lambda| \equiv |\alpha| \pmod 5$, so the 5-core $|\alpha|$ has the form $5k+4$ for some k with $k \leq m$. Let $c(n)$ be the number of 5-cores of weight n. Thus, the generating function for $c(n)$ is $G_5(x)$. From Eq. (7), we get the congruence

$$\sum_{n\geq 0} c(n)x^n = \frac{\prod_{n\geq 1}(1-x^{5n})^5}{\prod_{n\geq 1}(1-x^n)} \tag{9}$$

$$\equiv \frac{\prod_{n\geq 1}(1-x^{n})^{25}}{\prod_{n\geq 1}(1-x^n)} \pmod 5 \tag{10}$$

$$\equiv \prod_{n\geq 1}(1-x^n)^{24} \pmod 5. \tag{11}$$

Recall that

$$\sum_{n\geq 0} \tau(n)x^{n-1} = \prod_{n\geq 1}(1-x^n)^{24},$$

where τ is the Ramanujan τ function. (See [4] for a wealth of information about τ and related functions.) From the recurrence relations for τ on p. 164 of [4], it follows that $5|\tau(5t)$ for all positive integers t. In particular, $5|\tau(5k+5)$, so we get $5|c(5k+4)$ from lines (9)–(11). By grouping together trees that share the same 5-quotient, it follows that $5|c(5k+4)$ implies $5|p(5m+4)$. Thus, in the setting of tree representations of partitions, the root vertex of the tree plays the main role in Ramanujan's congruence (8), whereas the subtrees that form the quotient play a subsidiary role.

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