Europ. J. Combinatorics (1999) **20**, 101–114
Article No. eujc.1998.0272
Available online at http://www.idealibrary.com on IDEAL

Upper Bounds on Permutation Codes via Linear Programming

HANNU TARNANEN

An upper bound on permutation codes of length n is given. This bound is a solution of a certain linear programming problem and is based on the well-developed theory of association schemes. Several examples are presented. For instance, the 255 values of the bound for $n \leq 8$ are tabulated. It turns out that, for $n \leq 8$, the Kiyota bound for group codes also holds for unrestricted codes at least in 178 cases. Also an easier (but weaker) polynomial version of the bound is given. It is obtained by showing that the mappings $F_k(\theta)$ $(0 \leq k \leq n/2)$, where F_k is the Charlier polynomial of degree k and θ the natural character of the symmetric group S_n, are mutually orthogonal characters of S_n.

1. INTRODUCTION

Let n be a positive integer and denote by S_n the symmetric group of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and by $|X|$ the cardinality of a finite set X. Hence S_n is composed of all $n! = 1 \cdot 2 \cdot \ldots \cdot n$ permutations of $\mathbb{N}_n$, that is, of all bijections $\alpha : \mathbb{N}_n \to \mathbb{N}_n$, the product $\alpha\beta$ in S_n is the composite map $(\alpha\beta)(x) = \alpha(\beta(x))$ and the identity element of S_n is the identity permutation $1 : x \mapsto x$ of $\mathbb{N}_n$. The mapping

$$\theta(\alpha) = |\{x \in \mathbb{N}_n \mid \alpha(x) = x\}|$$

of S_n, which counts the number of fixed points of a given permutation α in S_n, is called the *natural character* of S_n. Thus $\theta(1) = n$ and $0 \leq \theta(\alpha) \leq n - 2$ for all α in $S_n \setminus \{1\}$, since no permutation of $\mathbb{N}_n$ has exactly $n - 1$ fixed points. The condition

$$\delta(\alpha, \beta) = n - \theta(\alpha^{-1}\beta) = |\{x \in \mathbb{N}_n \mid \alpha(x) \neq \beta(x)\}| \tag{1}$$

defines a metric on S_n. In fact, if a permutation $\alpha \in S_n$ is interpreted as a vector $\alpha = (\alpha(1), \ldots, \alpha(n))$, then $\delta(\alpha, \beta)$ $(\alpha, \beta \in S_n)$ is the Hamming distance between vectors α and β, that is, the number of component places in which the vectors α and β differ. Nonempty subsets of S_n are called *permutation codes of length n*. Let D be a subset of the ring $\mathbb{Z}$ of integers. A nonempty subset C of S_n is said to be a *D-code* in S_n if $\theta(\alpha^{-1}\beta) \in D$ whenever α and β are distinct elements of C. Denote by $M(n, D)$ the maximum cardinality of such a D-code. Then $M(n, \emptyset) = 1$, $M(n, \mathbb{Z}) = n!$ and $M(n, D) = M(n, D \cap \{0, 1, \ldots, n-2\})$ since $0 \leq \theta(\alpha^{-1}\beta) \leq n - 2$ for all distinct permutations α and β of $\mathbb{N}_n$.

In this paper an upper bound on $M(n, D)$ is given. This bound is a solution of a certain linear programming problem and is based on the well-developed theory of association schemes. Sections 2 and 3 are preliminary: we introduce association schemes, characters of symmetric groups and the conjugacy scheme which seems to be a suitable frame for studying permutation codes. In Section 4 the linear programming bound for $M(n, D)$ is given and several examples, both numerical and theoretical, are presented. In Section 5 we consider characters of S_n associated with the Charlier polynomials $F_k(x)$ of degree k in the real variable x. It is shown that the mappings $F_k(\theta)$ $(0 \leq k \leq n/2)$ are mutually orthogonal characters of S_n. This yields an easier (but weaker) polynomial version of the linear programming bound. Finally, examples illustrating the method are given.

0195-6698/99/010101 + 14 $30.00/0

2. Association Schemes

Let m be a nonnegative integer and R a *symmetric association scheme* with m classes on a finite nonempty set X having cardinality v. Hence R is a collection of $m+1$ symmetric relations $R_0, \ldots, R_m$ on X forming a partition of the cartesian power X^2, R_0 is the *diagonal relation* $\{(x,x) \mid x \in X\}$ of X and, for any triple of integers $i, j, k = 0, \ldots, m$, the cardinality

$$p_{ijk} = |\{z \in X \mid (x,z) \in R_i, (z,y) \in R_j\}|$$

is independent of the choice of $(x,y) \in R_k$. The numbers p_{ijk} are called the *intersection numbers* of the scheme. If $(x,y) \in R_i$, elements x and y are said to be *i-associates* in R. The positive integer p_{ii0} is called the *valence* of R_i and is denoted by v_i. By definition, v_i is the number of the i-associates of an arbitrary element of X. Denote by $\mathbb{R}(X,X)$ the algebra of all square matrices S of order v over the field $\mathbb{R}$ of real numbers, where the entries are numbered by the elements of X^2, the (x,y)-entry of S being written as $S(x,y)$. Let $\mathcal{B}$ be the linear subspace of $\mathbb{R}(X,X)$ generated by the *adjacency matrices* $A_0, \ldots, A_m \in \mathbb{R}(X,X)$ for which $A_i(x,y) = 1$ if $(x,y) \in R_i$ and $A_i(x,y) = 0$ if $(x,y) \notin R_i$. Then $\mathcal{B}$ is a commutative $(m+1)$-dimensional subalgebra of $\mathbb{R}(X,X)$ and is composed of symmetric matrices (cf. [9, p. 653]). This algebra is called the *Bose–Mesner algebra* of the scheme. It has a unique basis of *primitive idempotents* $J_0 = v^{-1}J, J_1, \ldots, J_m$ (J is the all-one matrix) which are nonzero matrices in $\mathcal{B}$ satisfying $J_i J_j = \delta_{ij} J_i$ where δ_{ij} is the Kronecker symbol (see [9, pp. 653 and 654]). Their ranks $\mu_i = \operatorname{rank} J_i$ are called the *multiplicities* of the scheme. Given the two bases $\{A_i\}$ and $\{J_i\}$ of $\mathcal{B}$, we have the basis transformations

$$A_k = \sum_{i=0}^{m} p_k(i) J_i \qquad \text{and} \qquad J_k = \frac{1}{v}\sum_{i=0}^{m} q_k(i) A_i \qquad (k = 0, \ldots, m).$$

Call the real coefficients $p_k(i)$ and $q_k(i)$ the *first* and *second eigennumbers* of the scheme. These parameters have the following properties (cf. [9, pp. 654 and 655]):

$$\sum_{i=0}^{m} p_k(i) q_i(r) = \sum_{i=0}^{m} q_k(i) p_i(r) = v\delta_{kr}, \tag{2}$$

$$p_0(i) = q_0(i) = 1, \qquad p_i(0) = v_i, \qquad q_i(0) = \mu_i, \tag{3}$$

$$v_i q_k(i) = \mu_k p_i(k). \tag{4}$$

Let D be a subset of the index set $\mathbb{N}_m$. A nonempty subset C of X is called a *D-clique* in R if any two distinct elements of C are i-associates for some i in D. Denote by $M(D)$ the maximum cardinality of a D-clique in R. Then $M(\emptyset) = 1$, $M(\mathbb{N}_m) = v$ and $M(D)$ is an increasing set function in the sense that $M(D) \le M(E)$ for $D \subseteq E \subseteq \mathbb{N}_m$.

The following theorem is called the *linear programming bound* (= LP bound) for cliques in R (see [3, Section 3.3.2]).

THEOREM 1. *Let D be a subset of $\mathbb{N}_m$. Subject to the constraints*

$$a_0 = 1, a_i \ge 0 \qquad \text{for } i \in \mathbb{N}_m, \tag{5}$$

$$a_i = 0 \qquad \text{for } i \in \mathbb{N}_m \setminus D, \tag{6}$$

$$a_0 q_k(0) + \cdots + a_m q_k(m) \ge 0 \qquad \text{for } k \in \mathbb{N}_m, \tag{7}$$

the sum $a_0 + \cdots + a_m$ has the maximum value $M_{LP}(D)$ and $M(D) \le M_{LP}(D)$.

Instead of the maximization problem of Theorem 1 it is sometimes useful to consider the dual minimization problem which is given in the next theorem (see [3, Section 3.3.2]).

THEOREM 2 (THE DUAL LP BOUND). *Let D be a subset of $\mathbb{N}_m$. If the mapping $F(i) = \beta_0 q_0(i) + \cdots + \beta_m q_m(i)$ of $\{0, \ldots, m\}$ satisfies the following two conditions*

$$\beta_0, \ldots, \beta_m \text{ are nonnegative real numbers} \qquad \text{and} \qquad \beta_0 > 0, \tag{8}$$

$$F(i) \leq 0 \qquad \text{for } i \in D, \tag{9}$$

then $M_{LP}(D) \leq F(0)/\beta_0$. Also, $M_{LP}(D) = F(0)/\beta_0$ for some mapping $F(i) = \beta_0 q_0(i) + \cdots + \beta_m q_m(i)$ of $\{0, \ldots, m\}$ satisfying conditions (8) *and* (9).

REMARKS. Evidently, the condition $\beta_0 > 0$ in Theorem 2 can be replaced by the condition $\beta_0 = 1$. Thus, according to (3), $M_{LP}(D)$ equals the minimum of the sum $1+\beta_1\mu_1+\cdots+\beta_m\mu_m$ subject to the constraints $\beta_i \geq 0$ ($i \in \mathbb{N}_m$) and $\beta_1 q_1(i)+\cdots+\beta_m q_m(i) \leq -1$ ($i \in D$). Hence $M_{LP}(D)$ can be found by linear programming if the second eigennumbers of the scheme are at disposal. Also, if a solution $\beta_1, \ldots, \beta_m$ of the above minimization problem is obtained by some numerical algorithm, then the validity of the bound $M(D) \leq 1 + \beta_1\mu_1 + \cdots + \beta_m\mu_m$ is easily verified by showing that the numbers $\beta_1, \ldots, \beta_m$ satisfy the constraints of the problem.

The following result is a slight improvement of a theorem of Delsarte [3, Theorem 3.9].

THEOREM 3. *$M_{LP}(D)M_{LP}(\mathbb{N}_m \setminus D) \leq v$ for $D \subseteq \mathbb{N}_m$.*

PROOF. Let $M_{LP}(D) = a_0 + \cdots + a_m$ where the numbers a_i satisfy the conditions (5), (6) and (7). Consider the map $F(i) = \beta_0 q_0(i) + \cdots + \beta_m q_m(i)$ where

$$\beta_k = \frac{1}{\mu_k} \sum_{i=0}^{m} a_i q_k(i). \tag{10}$$

By (7) and (3), $\beta_k \geq 0$ for $k \in \mathbb{N}_m$ and $\beta_0 = a_0+\cdots+a_m = M_{LP}(D) > 0$. According to (10), (4) and (2), $F(i) = a_i v/v_i$ for $i = 0, \ldots, m$. By (5), (3) and (6), $F(0) = v$ and $F(i) = 0$ for $i \in \mathbb{N}_m \setminus D$. Hence Theorem 2 yields the bound $M_{LP}(\mathbb{N}_m \setminus D) \leq F(0)/\beta_0 = v/M_{LP}(D)$. □

3. ON CHARACTERS OF THE SYMMETRIC GROUP

Let $\chi_0, \ldots, \chi_m$ be the distinct irreducible complex characters of S_n numbered such that $\chi_0 : \alpha \mapsto 1$ is the unit character. Further, let $d_k = \chi_k(1)$ be the *degree* of χ_k. Hence $d_1, \ldots, d_m$ are positive integers, $d_0 = 1$ and

$$d_0^2 + \cdots + d_m^2 = |S_n| = n! \tag{11}$$

(see [6, Corollary 2.7]). Recall that the values of χ_i are all integers (see [7, Theorem 1.2.17]). It is well-known (see [6, Corollary 2.7] and [7, Lemma 1.2.8]) that the number of distinct conjugacy classes $C(\alpha) = \{\beta\alpha\beta^{-1} \mid \beta \in S_n\}$ ($\alpha \in S_n$) of S_n equals the number $m + 1$ of irreducible complex characters of S_n and S_n is *ambivalent* in the sense that $C(\alpha^{-1}) = C(\alpha)$ for all $\alpha \in S_n$. Let $C_i = C(\alpha_i)$ ($i = 0, \ldots, m$) be the distinct conjugacy classes of S_n numbered such that $\alpha_0 = 1$. Hence $C_0 = \{1\}$.

THEOREM 4. *The relations*

$$R_i = \{(\alpha, \beta) \in S_n^2 \mid \alpha^{-1}\beta \in C_i\} \quad (i = 0, 1, \ldots, m)$$

form a symmetric association scheme with m classes on S_n (called the conjugacy scheme of S_n). The valences v_i and second eigennumbers $q_k(i)$ of this scheme are $v_i = |C_i|$ and $q_k(i) = d_k\chi_k(\alpha_i)$.

PROOF. Since S_n is ambivalent, the relations R_i are symmetric. The assertion follows from [3, Example 2.4.2]. □

Denote by $\mathrm{Cf}(S_n)$ the set of all real valued class functions of S_n. Hence a map $\varphi : S_n \to \mathbb{R}$ belongs to $\mathrm{Cf}(S_n)$ iff $\varphi(\alpha\beta\alpha^{-1}) = \varphi(\beta)$ holds for all elements α and β in S_n. Consider the set $\mathrm{Cf}(S_n)$ as a real linear algebra where the operations are defined pointwise and equip $\mathrm{Cf}(S_n)$ with the inner product

$$\langle\varphi, \psi\rangle_n = \frac{1}{n!}\sum_{\alpha \in S_n} \varphi(\alpha)\psi(\alpha).$$

It is well-known (cf. [6, Theorem 2.8 and Corollary 2.14]) that the characters $\chi_0, \ldots, \chi_m$ constitute an orthonormal basis of $\mathrm{Cf}(S_n)$. Thus each mapping φ in $\mathrm{Cf}(S_n)$ has a unique basis representation $\varphi = \beta_0\chi_0 + \cdots + \beta_m\chi_m$ where the β_i are real numbers and we have

$$\beta_i = \langle\varphi, \chi_i\rangle_n \qquad (i = 0, 1, \ldots, m). \tag{12}$$

We call these numbers β_i the *character coefficients of* φ. In particular, β_0 is called the *leading character coefficient* of φ. A nonzero map in $\mathrm{Cf}(S_n)$ is a character of S_n iff all its character coefficients are nonnegative integers (cf. [6, p. 15]).

Let $\cdot$ be an *action* of S_n on a finite nonempty set Ω. Hence $\cdot$ is a mapping $S_n \times \Omega \to \Omega$, $(\alpha, x) \mapsto \alpha x$ satisfying the conditions $1x = x$ and $\alpha(\beta x) = (\alpha\beta)x$ for all $\alpha, \beta \in S_n$ and $x \in \Omega$. The map $\pi(\alpha) = |\{x \in \Omega \mid \alpha x = x\}|$ of S_n is a character of S_n (see [6, p. 68]) called the *permutation character* of S_n associated with the action. The action and the associated permutation character are called *transitive* if, for all x and y in Ω, there exists a permutation $\alpha \in S_n$ such that $\alpha x = y$. It is well-known (cf. [6, Corollary 5.15]) that $\langle\pi, \chi_0\rangle_n$ equals the number of the distinct *orbits* $S_n x = \{\alpha x \mid \alpha \in S_n\}$ $(x \in \Omega)$ of the action. Hence π is transitive iff $\langle\pi, \chi_0\rangle_n = 1$. Let $r \le |\Omega|$ be a positive integer. A vector $(x_1, \ldots, x_r)$ is called an *r-permutation* of Ω if its components $x_1, \ldots, x_r$ are distinct elements of Ω. Denote by $\Omega^{(r)}$ the set of all r-permutations of Ω. Evidently, the condition $\alpha(x_1, \ldots, x_r) = (\alpha x_1, \ldots, \alpha x_r)$ defines an action of S_n on $\Omega^{(r)}$ and the associated permutation character is

$$\pi_r = \prod_{k=0}^{r-1}(\pi - k\chi_0). \tag{13}$$

If π_r is transitive then the action $\cdot$ and character π are called *r-transitive*. Thus r-transitive $(r \ge 2)$ permutation characters are $(r-1)$-transitive. Further, π is 2-transitive iff $\pi = \chi_0 + \chi_i$ for some $i \ge 2$. (see [6, Corollary 5.17]). For example, the *natural action* $\alpha x = \alpha(x)$ of S_n on $\mathbb{N}_n$ is n-transitive and the associated permutation character is the natural character θ of S_n. In the case $n \ge 2$, we number the characters $\chi_0, \ldots, \chi_m$ such that $\theta = \chi_0 + \chi_1$.

A vector $p = (p_1, \ldots, p_n)$ of nonnegative integers is called a *partition* of n if $p_1 \ge p_2 \ge \cdots \ge p_n$ and $p_1 + p_2 + \cdots + p_n = n$. Let P_n be the set of all partitions of n. A partition $(p_1, \ldots, p_r, 0, \ldots, 0) \in P_n$ is also denoted by $(p_1, \ldots, p_r)$. The *conjugate* of a partition $p = (p_1, \ldots, p_n) \in P_n$ is defined to be the vector $p^* = (p_1^*, \ldots, p_n^*)$ of nonnegative integers $p_i^* = |\{j \in \mathbb{N}_n \mid p_j \geq i\}|$. Then $p^* \in P_n$ and $(p^*)^* = p$ for all partitions p of n. For $p = (p_1, \dots, p_n) \in P_n$, let Ω_p be the set of all vectors $(X_1, \dots, X_n)$ of subsets of $\mathbb{N}_n$ satisfying $X_1 \cup \cdots \cup X_n = \mathbb{N}_n$ and $|X_i| = p_i$ for $i = 1, \dots, n$. Then the condition $\alpha(X_1, \dots, X_n) = (\alpha(X_1), \dots, \alpha(X_n))$ defines a transitive action of S_n on Ω_p. Denote by π^p the permutation character of S_n associated with this action. By definition, $\pi^p(\alpha) = |\{X \in \Omega_p \mid \alpha X = X\}|$ for $p \in P_n$ and $\alpha \in S_n$. Hence $\pi^{(n)}$ is the unit character χ_0 and $\pi^{(n-1,1)}$ the natural character of S_n. Denote by sgn the *alternating character of* S_n defined by $\operatorname{sgn}(\alpha) = 1$ if $\alpha \in S_n$ is even, and $\operatorname{sgn}(\alpha) = -1$ if $\alpha \in S_n$ is odd. Since the products of characters are characters (see [6, Corollary 4.2]), then $\tau^p(\alpha) = \operatorname{sgn}(\alpha)\pi^p(\alpha)$ $(\alpha \in S_n)$ is a character of S_n for all $p \in P_n$. It is well-known (see [7, Section 2.1]) that the irreducible characters $\chi_0, \dots, \chi_m$ of S_n have a unique numbering χ^p $(p \in P_n)$, called the *Frobenius numbering*, such that $\langle \pi^p, \chi^p \rangle_n \neq 0$ and $\langle \tau^{p*}, \chi^p \rangle_n \neq 0$ for $p \in P_n$.

Consider S_n as the subgroup $\{\alpha \in S_{n+1} \mid \alpha(n+1) = n+1\}$ of S_{n+1}. Given a class function $\varphi \in \mathrm{Cf}(S_n)$, define the *induced function* $\varphi \uparrow$ of S_{n+1} by

$$(\varphi \uparrow)(\alpha) = \frac{1}{n!} \sum_{\beta \in S_{n+1}} \varphi(\beta\alpha\beta^{-1}) \qquad (\alpha \in S_{n+1}),$$

where we have set $\varphi(\gamma) = 0$ for $\gamma \notin S_n$. Then $\varphi \uparrow \in \mathrm{Cf}(S_{n+1})$ and $(\varphi \uparrow)(1) = (n+1)\varphi(1)$ for all $\varphi \in \mathrm{Cf}(S_n)$. Evidently, the mapping $\mathrm{Cf}(S_n) \to \mathrm{Cf}(S_{n+1})$, $\varphi \mapsto \varphi \uparrow$ is linear. Also, if χ is a character of S_n, then $\chi \uparrow$ is a character of S_{n+1} (see [6, Corollary 5.3]) called the *character of* S_{n+1} *induced by* χ. For example, $\chi_0 \uparrow$ is the natural character of S_{n+1} (see [6, Lemma 5.14]). Denote by $\varphi \downarrow$ the restriction of a class function $\varphi \in \mathrm{Cf}(S_n)$ to the group $S_{n-1} = \{\alpha \in S_n \mid \alpha(n) = n\}$. Then $\varphi \downarrow \in \mathrm{Cf}(S_{n-1})$ for $\varphi \in \mathrm{Cf}(S_n)$ and the mapping $\mathrm{Cf}(S_n) \to \mathrm{Cf}(S_{n-1})$, $\varphi \mapsto \varphi \downarrow$ is an algebra morphism. Evidently, $\chi \downarrow$ is a character of S_{n-1}, if χ is a character of S_n.

Given partitions $p = (p_1, \dots, p_n) \in P_n$ and $q \in P_{n+1}$, denote $p < q$ or $q > p$ if the vectors $(p_1, \dots, p_n, 0)$ and q differ in exactly one component place. Then, by the *Schur's branching law* (see [7, Theorem 2.4.3]), for all $p \in P_n$, we have

$$\chi^p \uparrow = \sum_{\substack{q \in P_{n+1} \\ q > p}} \chi^q \qquad \text{and} \qquad \chi^p \downarrow = \sum_{\substack{q \in P_{n-1} \\ q < p}} \chi^q. \tag{14}$$

4. Linear Programming Bounds for Permutation Codes

As in Section 3, let $\chi_0, \dots, \chi_m$ be the distinct irreducible complex characters of S_n and $C_i = C(\alpha_i)$ $(i = 0, \dots, m)$ the distinct conjugacy classes of S_n numbered such that χ_0 is the unit character, $\alpha_0 = 1$ and, for $n \geq 2$, $\theta = \chi_0 + \chi_1$. Further, let D be a set of integers.

Since the natural character θ is a class function, then the D-codes in S_n are the $\{i \in \mathbb{N}_m \mid \theta(\alpha_i) \in D\}$-cliques in the conjugacy scheme of S_n. Hence Theorems 1 and 4 yield

THEOREM 5 (THE LP BOUND FOR PERMUTATION CODES). *Subject to the constraints*

$$a_0 = 1, a_i \geq 0 \qquad \text{for } i \in \mathbb{N}_m, \tag{15}$$
$$a_i = 0 \qquad \text{for } i \in \mathbb{N}_m \text{ such that } \theta(\alpha_i) \in \mathbb{N}_m \setminus D, \tag{16}$$
$$a_0\chi_k(\alpha_0) + \cdots + a_m\chi_k(\alpha_m) \geq 0 \qquad \text{for } k \in \mathbb{N}_m, \tag{17}$$

the sum $a_0 + \cdots + a_m$ *has the maximum value* $M_{LP}(n, D)$ *and* $M(n, D) \leq M_{LP}(n, D)$.

A class function $\varphi \in \mathrm{Cf}(S_n)$ is *nonnegative* if $\langle \varphi, \chi_i \rangle_n \geq 0$ for $i = 0, \ldots, m$. Call a nonnegative class function $\varphi \in \mathrm{Cf}(S_n)$ a *D-program of* S_n if $\langle \varphi, \chi_0 \rangle_n > 0$ and $\varphi(\alpha_i) \leq 0$ for all $i \in \mathbb{N}_m$ satisfying $\theta(\alpha_i) \in D$.

Identify a real valued map F of $\{0, 1, \ldots, m\}$ with the class function $\varphi \in \mathrm{Cf}(S_n)$ defined by $\varphi(\alpha) = F(i)$ for $\alpha \in C_i$. Then (12) and Theorems 2 and 4 yield

THEOREM 6 (THE DUAL LP BOUND FOR PERMUTATION CODES). *If φ is a D-program of S_n, then*

$$M_{LP}(n, D) \leq \varphi(1)/\langle \varphi, \chi_0 \rangle_n.$$

Also, $M_{LP}(n, D) = \varphi(1)/\langle \varphi, \chi_0 \rangle_n$ for some D-program φ of S_n.

REMARKS. The maximum value $M_{LP}(n, D)$ can be found by linear programming if the character table of S_n is at disposal. For example, in [7, pp. 348–355] these tables are given for $n \leq 10$. Denote

$$\Pi_n(D) = \prod_{i \in D'} (n - i),$$

where $D' = D \cap \{0, 1, \ldots, n-2\}$. Kiyota [8] has shown that if a subgroup G of S_n is a D-code in S_n, then $|G|$ divides $\Pi_n(D)$ and so $|G| \leq \Pi_n(D)$. The determination of sets D, for which

$$M(n, D) \leq \Pi_n(D) \tag{18}$$

holds, is an open problem (see [1, p. 36]). According to Theorem 3,

$$M_{LP}(n, D) M_{LP}(n, D^c) \leq n! \tag{19}$$

where $D^c = \mathbb{Z} \setminus D$. Since $\Pi_n(D)\Pi_n(D^c) = n!$ then, by (19), the bound (18) always holds for $M(n, D)$ or for $M(n, D^c)$.

EXAMPLE 1. Given an integer d with $2 \leq d \leq n$, call a nonempty subset C of S_n a *d-code in S_n* if $\delta(\alpha, \beta) \geq d$ whenever α and β are distinct elements of C. Denote by $M(n, d)$ the maximum cardinality of a d-code in S_n. According to (1), $M(n, d) = M(n, \{0, 1, \ldots, n-d\})$.

Let $r \leq n$ be a positive integer. Since, by (13),

$$\theta_r = \prod_{k=0}^{r-1} (\theta - k\chi_0)$$

is a transitive permutation character of S_n, then θ_r is nonnegative and $\langle \theta_r, \chi_0 \rangle_n = 1$. We also have $\theta_r(\alpha) = 0$ for all $\alpha \in S_n$ such that $\theta(\alpha) \leq r-1$. Hence θ_{n-d+1} is a $\{0, \ldots, n-d\}$-program of S_n and Theorem 6 gives the bound of Blake *et al.* [2]:

$$M(n, d) \leq d(d+1)\ldots n. \tag{20}$$

Hence $M(n, d)$ satisfies (18) for all positive integers n and d with $2 \leq d \leq n$.

EXAMPLE 2. The simplex algorithm yields that, for $n \leq 6$, we have $M_{LP}(n, D) = \Pi_n(D)$ with the exception of the following 20 cases:

n	D	$M_{LP}(n, D)$	$\Pi_n(D)$	$M_{LP}(n, D^c)$	$\Pi_n(D^c)$
5	013	20	40	6	3
5	023	15	30	8	4
6	013	60	90	12	8
6	014	30	60	24	12
6	023	48	72	15	10
6	124	30	40	24	18
6	134	15	30	48	24
6	0124	120	240	6	3
6	0134	60	180	12	4
6	0234	48	144	15	5

where, as well as in Tables 1 and 2, the set $D = \{a_1, a_2, \ldots, a_r\}$ is given as a sequence $a_1 a_2 \ldots a_r$. Thus

$$M_{LP}(n, D) M_{LP}(n, D^c) = n!$$

for all $n \leq 6$ and $D \subseteq \mathbb{Z}$. For $n \leq 8$, Tables 1 and 2 show that in the totality of 255 cases the bound $M(n, D) \leq \Pi_n(D)$ holds at least in 178 cases. Also, for $n \leq 8$ and $D \subseteq \{0, 1, \ldots, n-2\}$, we have $M(n, D) \leq \Pi_n(D)$ if $|D| \geq n/2$ and (n, D) is none of the three pairs $(8, 0124)$, $(8, 0456)$ and $(8, 1456)$.

In Tables 1 and 2 the values $M_{LP}(n, D)$ were calculated solving the dual LP bound by Mathematica subroutine LinearProgramming (see [11, p. 819]) which operates with exact rational numbers. The results were tested by veryfying that the solution vectors were indeed D-programs. The required character tables were taken from [7, pp. 350 and 351] and, in addition to the manual checking, the copied data were tested by the orthogonality relations of the characters.

EXAMPLE 3. Denote by χ_{k-1} and C_{k-1} the character χ_k and conjugacy class C_k of S_n in the tables [7, pp. 349–355]. Seven out of the 15 irreducible characters of S_7 are given in the following table:

	χ_1	χ_2	χ_3	χ_4	χ_{11}	χ_{12}	χ_{13}
C_0	6	14	15	14	14	14	6
C_1	4	6	5	4	−4	−6	−4
C_2	2	2	−1	2	2	2	2
C_3	0	2	−3	0	0	−2	0
C_4	3	2	3	−1	−1	2	3
C_5	1	0	−1	1	−1	0	−1
C_6	−1	2	−1	−1	−1	2	−1
C_7	0	−1	0	2	2	−1	0
C_8	2	0	1	−2	2	0	−2
C_9	0	0	−1	0	0	0	0
C_{10}	−1	0	1	1	−1	0	1
C_{11}	1	−1	0	−1	−1	−1	1
C_{12}	−1	1	0	−1	1	−1	1
C_{13}	0	−1	0	0	0	1	0
C_{14}	−1	0	1	0	0	0	−1

TABLE 1.
The integer part M of $M_{LP}(7, D)$ and the values of $\Pi = \Pi_7(D)$.

D	M	Π	D	M	Π	D	M	Π	D	M	Π
∅	1	1	23	60	20	123	120	120	0234	93	420
0	7	7	24	15	15	124	108	90	0235	140	280
1	30	6	25	15	10	125	60	60	0245	63	210
2	15	5	34	26	12	134	72	72	0345	168	168
3	12	4	35	12	8	135	72	48	1234	360	360
4	8	3	45	9	6	145	54	36	1235	120	240
5	2	2	012	140	210	234	60	60	1245	108	180
01	42	42	013	205	168	235	60	40	1345	72	144
02	52	35	014	84	126	245	15	30	2345	120	120
03	46	28	015	84	84	345	32	24	01234	2520	2520
04	42	21	023	93	140	0123	543	840	01235	630	1680
05	14	14	024	63	105	0124	420	630	01245	420	1260
12	30	30	025	52	70	0125	172	420	01345	280	1008
13	72	24	034	84	84	0134	280	504	02345	168	840
14	36	18	035	46	56	0135	205	336	12345	720	720
15	48	12	045	42	42	0145	84	252	012345	5040	5040

TABLE 2.
The integer part M of $M_{LP}(8, D)$ and the values of $\Pi = \Pi_8(D)$.

D	M	Π	D	M	Π	D	M	Π	D	M	Π
∅	1	1	015	224	168	0123	926	1680	2356	180	180
0	8	8	016	112	112	0124	1489	1344	2456	104	144
1	42	7	023	192	240	0125	584	1008	3456	120	120
2	42	6	024	192	192	0126	403	672	01234	4135	6720
3	15	5	025	226	144	0134	625	1120	01235	2520	5040
4	13	4	026	224	96	0135	373	840	01236	1032	3360
5	8	3	034	160	160	0136	280	560	01245	1792	4032
6	2	2	035	120	120	0145	224	672	01246	1489	2688
01	56	56	036	112	80	0146	270	448	01256	660	2016
02	192	48	045	211	96	0156	287	336	01345	625	3360
03	100	40	046	96	64	0234	192	960	01346	695	2240
04	96	32	056	67	48	0235	330	720	01356	373	1680
05	64	24	123	147	210	0236	264	480	01456	395	1344
06	16	16	124	253	168	0245	432	576	02345	480	2880
12	42	42	125	177	126	0246	384	384	02346	384	1920
13	75	35	126	84	84	0256	226	288	02356	338	1440
14	108	28	134	175	140	0345	480	480	02456	478	1152
15	105	21	135	105	105	0346	226	320	03456	960	960
16	70	14	136	79	70	0356	120	240	12345	2520	2520
23	102	30	145	152	84	0456	261	192	12346	630	1680
24	104	24	146	112	56	1234	543	840	12356	420	1260
25	58	18	156	136	42	1235	420	630	12456	403	1008
26	64	12	234	138	120	1236	188	420	13456	192	840
34	60	20	235	135	90	1245	360	504	23456	720	720
35	15	15	236	180	60	1246	253	336	012345	20160	20160
36	15	10	245	104	72	1256	252	252	012346	5040	13440
45	38	12	246	104	48	1345	180	420	012356	2520	10080
46	13	8	256	64	36	1346	175	280	012456	1792	8064
56	9	6	345	96	60	1356	136	210	013456	960	6720
012	336	336	346	62	40	1456	190	168	023456	960	5760
013	280	280	356	15	30	2345	360	360	123456	5040	5040
014	224	224	456	43	24	2346	180	240	0123456	40320	40320

where the (C_i, χ_j) entry gives the value $\chi_j(\alpha_i)$ of χ_j in the conjugacy class C_i and $\theta = \chi_0 + \chi_1$. Hence θ has the inverse images $\theta^{-1}(0) = C_6 \cup C_{10} \cup C_{12} \cup C_{14}$, $\theta^{-1}(1) = C_3 \cup C_7 \cup C_9 \cup C_{13}$ and $\theta^{-1}(2) = C_5 \cup C_{11}$.

For $\varphi = 6\chi_0 + 21\chi_1 + 14\chi_2 + 18\chi_3 + \chi_4 + 7\chi_{11} + 8\chi_{12} + 3\chi_{13}$, we have $\varphi(1) = 840$, $\varphi(\alpha_3) = -36$, $\varphi(\alpha_9) = -12$ and $\varphi(\alpha_i) = 0$ $(i = 6, 10, 12, 14, 7, 13, 5, 11)$. Thus φ is a $\{0, 1, 2\}$-program of S_7 and $M(7, 5) \leq 140$. Similarly, solving the dual LP bound by linear programming, one obtains the following improvements of (20):

n	d	$M(n, d) \leq$	$d(d+1)\dots n$
7	4	543	840
7	5	140	210
8	4	4135	6720
8	5	926	1680
9	4	32989	60480
9	5	7128	15120
9	6	1962	3024
10	4	302400	604800
10	5	64800	151200
10	6	16941	30240
10	7	4699	5040

Hence the inequality $M(n, d) \leq nM(n-1, d)$ $(2 \leq d \leq n-1)$ (see [2]) yields the following bound: for $n \geq 10$ and $d = 4, 5, 6, 7$, we have

$$M(n, d) \leq C_d \cdot d(d+1)\dots n$$

where $C_4 = 1/2$, $C_5 = 3/7 \leq 0.4286$, $C_6 = 5647/10080 \leq 0.5603$ and $C_7 = 4699/5040 \leq 0.9324$.

LEMMA 1. *Denote $D\uparrow = \{d+1 \mid d \in D\}$. If φ is a D-program of S_n, then the induced mapping $\varphi\uparrow$ is a $(\{0\} \cup D\uparrow)$-program of S_{n+1}, $(\varphi\uparrow)(1) = (n+1)\varphi(1)$ and the leading character coefficient of $\varphi\uparrow$ equals $\langle\varphi, \chi_0\rangle_n$.*

PROOF. We use the Frobenius numbering of the irreducible characters of S_n and S_{n+1}. Since φ is a D-program then

$$\varphi = \sum_{p \in P_n} \beta_p \chi^p,$$

where $\beta_p = \langle\varphi, \chi^p\rangle_n \geq 0$ and $\beta_{(n)} = \langle\varphi, \chi_0\rangle_n > 0$. By Schur's branching law (14),

$$\varphi\uparrow = \sum_{q \in P_{n+1}} \left(\sum_{p<q} \beta_p \right) \chi^q.$$

Hence $\varphi\uparrow$ is nonnegative and the leading character coefficient of $\varphi\uparrow$ is

$$\sum_{p<(n+1)} \beta_p = \beta_{(n)} = \langle\varphi, \chi_0\rangle_n > 0.$$

Let θ' be the natural character of S_{n+1}. Suppose $\alpha \in S_{n+1} \setminus \{1\}$ and $\theta'(\alpha) = d+1$ where $d \in D$. If $\beta \in S_{n+1}$ and $\beta\alpha\beta^{-1} \in S_n$, then $n+1$ is a fixed point of $\beta\alpha\beta^{-1}$ and hence $\theta(\beta\alpha\beta^{-1}) = \theta'(\beta\alpha\beta^{-1}) - 1 = \theta'(\alpha) - 1 = d \in D$. Consequently, $(\varphi\uparrow)(\alpha) \leq 0$. On the other hand, if a permutation $\alpha \in S_{n+1}$ has no fixed points then $(\varphi\uparrow)(\alpha) = 0$, since the conditions $\beta \in S_{n+1}$ and $\beta\alpha\beta^{-1} \in S_n$ imply $\alpha(\beta^{-1}(n+1)) = \beta^{-1}(n+1)$. Thus $\varphi\uparrow$ is a $(\{0\} \cup D\uparrow)$-program of S_{n+1} and the proof is complete. □

EXAMPLE 4. Use a D-program φ of S_n satisfying $M_{LP}(n, D) = \varphi(1)/\langle\varphi, \chi_0\rangle_n$. Then Lemma 1 yields the inequality

$$M_{LP}(n+1, \{0\} \cup D\uparrow) \leq (n+1)M_{LP}(n, D).$$

Also, since $\Pi_{n+1}(\{0\} \cup D\uparrow) = (n+1)\Pi_n(D)$, then

$$M_{LP}(n+1, \{0\} \cup D\uparrow) \leq \Pi_{n+1}(\{0\} \cup D\uparrow)$$

provided $M_{LP}(n, D) \leq \Pi_n(D)$. For instance, Table 2 gives 80 subsets D of the set $\{0, 1, \ldots, 7\}$ for which the inequality $M_{LP}(9, D) \leq \Pi_9(D)$ holds.

LEMMA 2. *Denote* $D\downarrow = \{d-1 \mid d \in D\}$. *If* φ *is a* D*-program of* S_n, *where* $n \geq 2$, *then* $\varphi\downarrow$ *is a* $(D\downarrow)$*-program of* S_{n-1} *and the leading character coefficient of* $\varphi\downarrow$ *equals* $\langle\varphi, \theta\rangle_n$.

PROOF. Since φ is a D-program then

$$\varphi = \sum_{p \in P_n} \beta_p \chi^p,$$

where $\beta_p = \langle\varphi, \chi^p\rangle_n \geq 0$ and $\beta_{(n)} = \langle\varphi, \chi_0\rangle_n > 0$. By (14),

$$\varphi\downarrow = \sum_{q \in P_{n-1}} \left(\sum_{p>q} \beta_p\right) \chi^q.$$

Hence $\varphi\downarrow$ is nonnegative and the leading character coefficient of $\varphi\downarrow$ is

$$\sum_{p>(n-1)} \beta_p = \beta_{(n)} + \beta_{(n-1,1)} = \langle\varphi, \chi_0\rangle_n + \langle\varphi, \chi_1\rangle_n = \langle\varphi, \theta\rangle_n > 0.$$

Let θ' be the natural character of S_{n-1}. Suppose $\alpha \in S_{n-1} \setminus \{1\}$ and $\theta'(\alpha) = d-1$ where $d \in D$. Since $\alpha(n) = n$, then $\theta(\alpha) = \theta'(\alpha) + 1 = d \in D$ and $(\varphi\downarrow)(\alpha) = \varphi(\alpha) \leq 0$. Thus $\varphi\downarrow$ is a $(D\downarrow)$-program of S_{n-1}. □

LEMMA 3. $M_{LP}(n-1, D) \leq M_{LP}(n, D\uparrow)$ *for* $n \geq 2$.

PROOF. Let φ be a $(D\uparrow)$-program of S_n such that $M_{LP}(n, D\uparrow) = \varphi(1)/\langle\varphi, \chi_0\rangle_n$. By Lemma 2, $\varphi\downarrow$ is a D-program of S_{n-1} and its leading character coefficient is $\langle\varphi, \theta\rangle_n = \langle\varphi, \chi_0\rangle_n + \langle\varphi, \chi_1\rangle_n \geq \langle\varphi, \chi_0\rangle_n$. Consequently,

$$M_{LP}(n-1, D) \leq (\varphi\downarrow)(1)/\langle\varphi, \theta\rangle_n \leq \varphi(1)/\langle\varphi, \chi_0\rangle_n = M_{LP}(n, D\uparrow).$$ □

EXAMPLE 5. Suppose $n \geq 2$ and $D \subseteq \{0, 1, \ldots, n-3\}$. Then $\Pi_n(D\uparrow) = \Pi_{n-1}(D)$ and hence $M_{LP}(n, D\uparrow) \geq \Pi_n(D\uparrow)$ and $M_{LP}(n, (D\uparrow)^c) \leq \Pi_n((D\uparrow)^c)$ provided $M_{LP}(n-1, D) \geq \Pi_{n-1}(D)$. For instance, Table 2 gives 76 subsets D of $\{0, 1, \ldots, 7\}$ satisfying $M_{LP}(9, D) \leq \Pi_9(D)$.

5. CHARACTERS ASSOCIATED WITH CHARLIER POLYNOMIALS

Let w_k be the number of elements in S_n having exactly k fixed points (see [4]):

$$w_k = |\{\alpha \in S_n \mid \theta(\alpha) = k\}| = \frac{n!}{k!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} \qquad (k = 0, \ldots, n).$$

Note that $w_{n-1} = 0$ and all the other numbers w_k $(k = 0, \dots, n, k \neq n-1)$ are positive. Equip the algebra $\mathbb{R}[x]$ of all real polynomials in the variable x with the symmetric bilinear form

$$(f, g)_n = \frac{1}{n!} \sum_{k=0}^{n} w_k f(k) g(k). \tag{21}$$

Then (21) defines an inner product on the n-dimensional vector space $\mathbb{R}_n[x]$ of all real polynomials of degree $n-1$ or less in the variable x. Associate with each polynomial $f(x) = a_r x^r + \cdots + a_1 x + a_0$ $(a_i \in \mathbb{R})$ in $\mathbb{R}[x]$ the class function $f(\theta) = a_r \theta^r + \cdots + a_1 \theta + a_0 \chi_0$ in $\mathrm{Cf}(S_n)$. Evidently, the mapping $\mathbb{R}[x] \to \mathrm{Cf}(S_n)$, $f \mapsto f(\theta)$ is an algebra morphism and $f(\theta)(\alpha) = f(\theta(\alpha))$ holds for all $f \in \mathbb{R}[x]$ and $\alpha \in S_n$. Hence

$$\langle f(\theta), g(\theta) \rangle_n = (f, g)_n \qquad (f, g \in \mathbb{R}[x]). \tag{22}$$

In particular, the leading character coefficient of $f(\theta)$ is

$$\langle f(\theta), \chi_0 \rangle_n = (f, 1)_n \qquad (f \in \mathbb{R}[x]).$$

Our aim is to upper bound $M(n, D)$ by using polynomial functions of θ associated with the *Charlier polynomials* $F_k = F_k(x)$ $(k = 0, 1, 2, \dots)$ defined by

$$F_k(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} x_{(i)},$$

where $x_{(0)} = 1$ and $x_{(i)} = x(x-1)\dots(x-i+1)$ for $i \geq 1$. Evidently, $F_k(x)$ is a monic polynomial of degree k,

$$F_0(x) = 1 \qquad \text{and} \qquad F_1(x) = x - 1. \tag{23}$$

We also have the three-term recurrence relation (cf. [5, p. 163])

$$F_{k+1}(x) = (x - k - 1) F_k(x) - k F_{k-1}(x) \qquad (k = 1, 2, \dots). \tag{24}$$

Hence

$$\begin{aligned}
F_2(x) &= x^2 - 3x + 1, \\
F_3(x) &= x^3 - 6x^2 + 8x - 1, \\
F_4(x) &= x^4 - 10x^3 + 29x^2 - 24x + 1, \\
F_5(x) &= x^5 - 15x^4 + 75x^3 - 145x^2 + 89x - 1.
\end{aligned}$$

LEMMA 4. *$\theta\varphi = (\varphi\downarrow)\uparrow$ for $\varphi \in \mathrm{Cf}(S_n)$.*

PROOF. It is well-known (see [10, p. 20]) that $(\eta\uparrow)\zeta = (\eta(\zeta\downarrow))\uparrow$ for all characters η of S_{n-1} and ζ of S_n. Since $\theta = \chi\uparrow$ where χ is the unit character of S_{n-1} then, for all characters ζ of S_n, we have $\theta\zeta = (\chi\uparrow)\zeta = (\chi(\zeta\downarrow))\uparrow = (\zeta\downarrow)\uparrow$. The assertion follows by the linearity of the mappings $\downarrow$ and $\uparrow$. □

Given a partition $p = (p_1, \dots, p_n)$ of n and an integer $r \geq n$, denote by d^p the degree $\chi^p(1)$ of χ^p and by (r, p) the partition $(r, p_1, \dots, p_n)$ of $r + n$.

THEOREM 7. *For $0 \leq k \leq n/2$, we have*

$$F_k(\theta) = \sum_{p \in P_k} d^p \chi^{(n-k, p)}. \tag{25}$$

PROOF. Denote by ξ_k the right-hand side of (25). According to (11), $d^{(0)} = d^{(1)} = 1$ and hence, by (23), $\xi_0 = d^{(0)}\chi^{(n)} = \chi_0 = F_0(\theta)$ and $\xi_1 = d^{(1)}\chi^{(n-1,1)} = \chi_1 = \theta - \chi_0 = F_1(\theta)$. By (24), it is sufficient to prove that

$$\xi_{k+1} = (\theta - k - 1)\xi_k - k\xi_{k-1} \qquad \text{for } 1 \leq k \leq (n-2)/2. \tag{26}$$

Suppose $1 \leq k \leq (n-2)/2$ and let p be a partition of k. By (14) and Lemma 4,

$$\begin{aligned}\theta\chi^{(n-k,p)} &= \left(\chi^{(n-k-1,p)} + \sum_{q<p}\chi^{(n-k,q)}\right)\uparrow \\ &= \chi^{(n-k,p)} + \sum_{q>p}\chi^{(n-k-1,q)} + \sum_{q<p}\chi^{(n-k+1,q)} + \sum_{q<p}\sum_{r>q}\chi^{(n-k,r)}.\end{aligned}$$

Hence

$$\theta\xi_k = \xi_k + A_1 + A_2 + A_3$$

where

$$\begin{aligned}A_1 &= \sum_{p\in P_k} d^p \sum_{q>p}\chi^{(n-k-1,q)} = \sum_{q\in P_{k+1}}\left(\sum_{p<q} d^p\right)\chi^{(n-k-1,q)}, \\ A_2 &= \sum_{q\in P_{k-1}}\left(\sum_{p>q} d^p\right)\chi^{(n-k+1,q)}\end{aligned}$$

and

$$A_3 = \sum_{r\in P_k}\left(\sum_{q<r}\sum_{p>q} d^p\right)\chi^{(n-k,r)}.$$

For $q \in P_{k+1}$, we have

$$d^q = (\chi^q \downarrow)(1) = \sum_{p<q}\chi^p(1) = \sum_{p<q} d^p$$

and hence $A_1 = \xi_{k+1}$. For $q \in P_{k-1}$, we have

$$kd^q = k\chi^q(1) = (\chi^q \uparrow)(1) = \sum_{p>q}\chi^p(1) = \sum_{p>q} d^p$$

and hence $A_2 = k\xi_{k-1}$. For $r \in P_k$, we have, by Lemma 4,

$$kd^r = ((\chi^r \downarrow)\uparrow)(1) = \sum_{q<r}\sum_{p>q}\chi^p(1) = \sum_{q<r}\sum_{p>q} d^p$$

and hence $A_3 = k\xi_k$. Consequently,

$$\theta\xi_k = \xi_k + A_1 + A_2 + A_3 = \xi_{k+1} + (k+1)\xi_k + k\xi_{k-1},$$

the recurrence relation (26) holds and the proof is complete. □

COROLLARY 1. *For $0 \leq r, s \leq n/2$, $F_r(\theta)$ is a character of S_n and*

$$\langle F_r(\theta), F_s(\theta)\rangle_n = (F_r, F_s)_n = r!\delta_{rs}.$$

PROOF. According to (22), (25) and (11),

$$(F_r, F_s)_n = \langle F_r(\theta), F_s(\theta)\rangle_n = \sum_{p\in P_r}\sum_{q\in P_s} d^p d^q \langle \chi^{(n-r,p)}, \chi^{(n-s,q)}\rangle_n$$
$$= \delta_{rs}\sum_{p\in P_r}(d^p)^2 = r!\delta_{rs}.$$

Since $\langle F_r(\theta), F_r(\theta)\rangle_n = r! \neq 0$, then $F_r(\theta)$ is nonzero. By (25), the character coefficients of $F_r(\theta)$ are nonnegative integers and so $F_r(\theta)$ is a character of S_n. □

COROLLARY 2. *Suppose* D *is a subset of* $\{0, 1, \ldots, n-2\}$ *and the polynomial* $F(x) = \beta_0 F_0(x) + \cdots + \beta_k F_k(x)$ $(1 \le k \le n/2)$ *satisfies the following two conditions:*

$$\beta_0, \ldots, \beta_k \text{ are nonnegative real numbers and } \beta_0 > 0. \tag{27}$$
$$F(i) \le 0 \text{ for } i \in D. \tag{28}$$

Then $M_{LP}(n, D) \le F(n)/\beta_0$.

PROOF. By Theorem 7 and Corollary 1, $F(\theta)$ is nonnegative and $\langle F(\theta), \chi_0\rangle_n = (F, F_0)_n = \beta_0 > 0$. Also if $\alpha \in S_n$ and $\theta(\alpha) \in D$, then $F(\theta)(\alpha) = F(\theta(\alpha)) \le 0$ by (28). Thus $F(\theta)$ is a D-program of S_n and Theorem 6 yields the bound $M_{LP}(n, D) \le F(\theta(1))/\beta_0 = F(n)/\beta_0$. □

EXAMPLE 6. Since $F_0 + F_2 = (x-1)(x-2)$, then

$$M(n, \{1, 2\}) \le (n-1)(n-2) \quad \text{for} \quad n \ge 4.$$

Since $F_0 + 3F_2 + F_3 = (x+1)(x-1)(x-3)$, then

$$M(n, \{1, 2, 3\}) \le (n+1)(n-1)(n-3) \quad \text{for} \quad n \ge 6.$$

Since $F_0 + F_3 = x(x-2)(x-4)$, then

$$M(n, \{0, 2, 3, 4\}) \le n(n-2)(n-4) \quad \text{for} \quad n \ge 6.$$

Since $3F_0 + 10F_2 + 8F_3 + 3F_4 = (x-1)(x-2)(x-4)(3x-1)$, then

$$M(n, \{1, 2, 3, 4\}) \le (n-1)(n-2)(n-4)(n-1/3) \quad \text{for} \quad n \ge 6.$$

Since $3F_0 + 4F_1 + F_4 = x(x-1)(x-4)(x-5)$, then

$$M(n, \{0, 1, 4, 5\}) \le n(n-1)(n-4)(n-5)/3 \quad \text{for} \quad n \ge 8.$$

Since $3F_0 + 5F_1 + 5F_4 + 3F_5 = x(x-1)(x-2)(x-4)(3x-19)$, then

$$M(n, \{0, 1, 2, 4, 5, 6\}) \le n(n-1)(n-2)(n-4)(n-19/3) \quad \text{for} \quad n \ge 10.$$

Since $11F_0 + 3F_2 + 5F_3 + 6F_4 = (x-1)(x-3)(x-5)(6x-1)$, then

$$M(n, \{1, 3, 4, 5\}) \le (n-1)(n-3)(n-5)(6n-1)/11 \quad \text{for} \quad n \ge 8.$$

Since $F_0 + F_1 + F_3 + F_4 = x(x-1)(x-3)(x-5)$, then

$$M(n, \{0, 1, 3, 4, 5\}) \le n(n-1)(n-3)(n-5) \quad \text{for} \quad n \ge 8.$$

Since $6F_0 + F_1 + 6F_3 + 2F_4 + F_5 = x(x-2)(x-3)^2(x-5)$, then

$$M(n, \{0, 2, 3, 4, 5\}) \le n(n-2)(n-3)^2(n-5)/6 \quad \text{for} \quad n \ge 10.$$

Since $11F_0 + 50F_2 + 65F_3 + 45F_4 + 11F_5 = (x-1)(x-2)(x-3)(x-5)(11x+1)$, then

$$M(n, \{1, 2, 3, 4, 5\}) \le (n-1)(n-2)(n-3)(n-5)(n+1/11) \quad \text{for} \quad n \ge 10.$$

If D is a fixed set of nonnegative integers, then $\Pi_n(D) \sim n^{|D|}$ as $n \to \infty$. Examples 1 and 6 give 24 subsets D of $\{0, 1, 2, 3, 4, 5\}$ satisfying $M(n, D) \lesssim n^{|D|}$ as $n \to \infty$.

REFERENCES

1. E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin-Cummings, Menlo Park, CA, 1984.
2. I. F. Blake, G. Cohen and M. Deza, Coding with permutations, *Inform. Control*, **43** (1979), 1–19.
3. P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Rep. Suppl.*, **10** (1973).
4. P. Frankl and M. Deza, On the maximum number of permutations with given maximal or minimal distance, *J. Comb. Theory Ser. A*, **22** (1977), 352–360.
5. C. D. Godsil, *Algebraic Combinatorics*, Chapman & Hall, New York, 1993.
6. I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.
7. G. D. James and A. Kerber, *The Representation Theory of Symmetric Groups*, Addison-Wesley, New York, 1981.
8. M. Kiyota, An inequality for finite permutation groups, *J. Comb. Theory Ser. A*, **27** (1979), 119.
9. F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, New York, 1977.
10. G. de B. Robinson, *Representation Theory of the Symmetric Group*, University of Toronto Press, Toronto, 1961.
11. S. Wolfram, *Mathematica, A System for Doing Mathematics by Computer*, Addison-Wesley, Redwood City, CA, 1991.

Received 20 December 1997 and accepted 13 October 1998

H. TARNANEN
Department of Mathematics,
University of Turku,
FIN-20014 Turku,
Finland