[25] J. Hagenauer, "Rate-compatible punctured convolutional codes (RCPC codes) and their applications," *IEEE Trans. Commun.*, vol. 36, pp. 389–400, Apr. 1988.
[26] Ø. Ytrehus, "A note on high rate binary convolutional codes," Univ. Bergen, Bergen, Norway, Tech. Rep. 68, 1992.
[27] I. E. Bocharova and B. D. Kudryashov, "Rational rate punctured convolutional codes for soft-decision viterbi decoding," *IEEE Trans. Inform. Theory*, vol. 43, pp. 1305–1313, July 1997.

Two Constructions of Permutation Arrays

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***Abstract*—In this correspondence, two new constructions of permutation arrays are given. A number of examples to illustrate the constructions are also provided.**

***Index Terms*—Bounds, code constructions, permutation arrays.**

I. Introduction

Let S_n denote the set of all $n!$ permutations of $Z_n = \{0, 1, \dots, n-1\}$. An (n, d) *permutation array* (PA) is a subset of S_n with the property that the Hamming distance between any two distinct permutations in the array is at least d. An (n, d) PA of size μ is called an (n, μ, d) PA.

PAs were somewhat studied in the 1970s, some important papers from that period are [2], [5], and [8]. A recent application by Vinck [14] of PAs to a coding/modulation scheme for communication over power lines has created renewed interest in PAs, see [3], [4], [6], [7], [10], [12], [15]–[17].

In this correspondence, we give a couple of new general constructions of PAs. Let $P(n, d)$ be the maximal size of an (n, d) PA. The constructions give improved lower bounds on $P(n, d)$ in some cases.

We say that a PA Π is *balanced* if for each position, each element of Z_n appears the same number of times in that position of the permutations of Π. In [6], we presented a construction of PAs using balanced PAs as building blocks. Under suitable conditions, the PAs constructed in this correspondence are balanced.

II. The Main Results

We use Z_q to denote the set $\{0, 1, \dots, q-1\}$ of integers, not the ring of integers modulo q. However, on a number of occasions, we do modular addition. For $k, l \in Z_q$, $k \oplus_q l$ denotes the unique integer in Z_q congruent to $k + l$ modulo q.

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Ordinary codes will be building blocks in our construction. A code C of length n over Z_q is a subset of Z_q^n. The minimum and maximum Hamming distances between distinct codewords of C are denoted by $d_{\min}(C)$ and $d_{\max}(C)$, respectively. A code of size M and minimum distance at least d is called an $(n, M, d; q)$ code, and a code of size M, minimum distance at least d_1 and maximum distance at most d_2 is called an $(n, M, d_1, d_2; q)$ code.

The maximal size M for which an $(n, M, d; q)$ code exists is denoted by $A_q(n, d)$ and the maximal size M for which an $(n, M, d_1, d_2; q)$ code exists is denoted by $A_q(n, d_1, d_2)$. Clearly, $A_q(n, d, n) = A_q(n, d)$. Tables of bounds on $A_q(n, d)$ are found in [1, pp. 463–498] for $q = 2$ and in [1, pp. 295–461] for general q (for linear codes). Bounds on $A_3(n, d)$ are given in [13].

We say that C is balanced if for each position, each element of Z_q appears the same number of times in that position of the codewords of C. In particular, a linear code over a finite field is balanced if there is no position where all the codewords are zero.

Codes are often defined over a finite field or a finite ring of size q, say. By renaming of the elements, we get a corresponding code over Z_q (algebraic properties may not be carried over, but this is not important in our context). In our examples, we therefore assume that the codes considered (e.g., Hamming codes) have elements in Z_q.

Definition 1: For $\boldsymbol{c} = [c_i]_{0 \le i < n} \in Z_q^n$, $\boldsymbol{\pi} = [\pi_i]_{0 \le i < n} \in S_n$, and $k \in Z_q$, let $\Lambda(\boldsymbol{c}, \boldsymbol{\pi}, k) = [\lambda_l]_{0 \le l < qn}$ where

$$\lambda_{jn+i} = (c_i \oplus_q j) \oplus_{qn} k \oplus_{qn} \pi_i q$$

for $0 \le i < n$ and $0 \le j < q$.

Example 1: Let $\boldsymbol{c} = 101$ and $\boldsymbol{\pi} = 120$. For $q = 2$ we get

$$\Lambda(\boldsymbol{c}, \boldsymbol{\pi}, 0) = 341250 \text{ and } \Lambda(\boldsymbol{c}, \boldsymbol{\pi}, 1) = 452301.$$

For $q = 3$ we get

k	0	1	2
$\Lambda(\boldsymbol{c}, \boldsymbol{\pi}, k)$	461572380	572683401	683704512

Remark: If $(c_i \oplus_q j) + k < q$ or $\pi_i < n-1$, then

$$(c_i \oplus_q j) \oplus_{qn} k \oplus_{qn} \pi_i q = (c_i \oplus_q j) + k + \pi_i q.$$

If $(c_i \oplus_q j) + k > q$ and $\pi_i = n-1$, then

$$(c_i \oplus_q j) \oplus_{qn} k \oplus_{qn} \pi_i q = (c_i \oplus_q j) + k - q.$$

Lemma 1: We have $\Lambda(\boldsymbol{c}, \boldsymbol{\pi}, k) \in S_{qn}$.

Proof: Since $\Lambda(\boldsymbol{c}, \boldsymbol{\pi}, k)$ has qn elements and they are all in Z_{qn} by definition, it remains to show that they are distinct. Hence, assume that

$$(c_i \oplus_q j) \oplus_{qn} k \oplus_{qn} \pi_i q = \left(c_{i'} \oplus_q j'\right) \oplus_{qn} k \oplus_{qn} \pi_{i'} q.$$

Then

$$(c_i \oplus_q j) + \pi_i q = \left(c_{i'} \oplus_q j'\right) + \pi_{i'} q$$

and so

$$\pi_i = \pi_{i'} \tag{1}$$

and

$$c_i \oplus_q j = c_{i'} \oplus_q j'. \tag{2}$$

Since $\boldsymbol{\pi}$ is a permutation, (1) implies that $i = i'$. Combined with (2) this implies that $j = j'$. QED

Theorem 1: Let C be an $(n, M, d; q)$ code and Π be an (n, μ, d) PA where $n \ge 2$. Let

$$d^* = \min\{qd, qn - (q-1)d_{\max}(C)\} \ge \min\{qd, n\}. \tag{3}$$

Then

i) $\Lambda(C,\Pi) \stackrel{\text{def}}{=} \{\Lambda(\boldsymbol{c},\boldsymbol{\pi},0) \mid \boldsymbol{c} \in C, \pi \in \Pi\}$ is a $(qn, M\mu, qd)$ PA;

ii) $\Lambda^*(C,\Pi) \stackrel{\text{def}}{=} \{\Lambda(\boldsymbol{c},\boldsymbol{\pi},k) \mid \boldsymbol{c} \in C, \pi \in \Pi, k \in Z_q\}$ is a $(qn, M\mu q, d^*)$ PA.

Moreover, if C and Π are balanced, then $\Lambda(C,\Pi)$ and $\Lambda^*(C,\Pi)$ are both balanced.

Proof: Let $\Lambda(\boldsymbol{c},\boldsymbol{\pi},k)$ and $\Lambda(\boldsymbol{c}',\boldsymbol{\pi}',k')$ be permutations. We will show that if $\boldsymbol{c} \neq \boldsymbol{c}'$, $\boldsymbol{\pi} \neq \boldsymbol{\pi}'$, or $k \neq k'$, then $\Lambda(\boldsymbol{c},\boldsymbol{\pi},k) \neq \Lambda(\boldsymbol{c}',\boldsymbol{\pi}',k')$. This implies the statements about the sizes of the PAs in i) and ii). Further, we will find lower bounds on $d_H(\Lambda(\boldsymbol{c},\boldsymbol{\pi},k), \Lambda(\boldsymbol{c}',\boldsymbol{\pi}',k'))$ when $\Lambda(\boldsymbol{c},\boldsymbol{\pi},k) \neq \Lambda(\boldsymbol{c}',\boldsymbol{\pi}',k')$ to prove the lower bounds on the minimum distances of the PAs in i) and ii).

We start by studying when pairs of corresponding elements in the two permutations are equal. Hence, suppose

$$(c_i \oplus_q j) \oplus_{qn} k \oplus_{qn} \pi_i q = \left(c'_i \oplus_q j\right) \oplus_{qn} k' \oplus_{qn} \pi'_i q. \tag{4}$$

First consider the case $k = k'$. Then (4) implies $(c_i \oplus_q j) + \pi_i q = (c'_i \oplus_q j) + \pi'_i q$ and so $\pi_i = \pi'_i$ and $c_i = c'_i$ (independent of j). Hence,

$$d_H(\Lambda(\boldsymbol{c},\boldsymbol{\pi},k), \Lambda(\boldsymbol{c}',\boldsymbol{\pi}',k)) = q|\{i \mid c_i \neq c'_i \text{ or } \pi_i \neq \pi'_i\}|.$$

This shows that

$$\text{if } \boldsymbol{c} \neq \boldsymbol{c}' \text{ or } \boldsymbol{\pi} \neq \boldsymbol{\pi}', \text{ then } \Lambda(\boldsymbol{c},\boldsymbol{\pi},k) \neq \Lambda(\boldsymbol{c}',\boldsymbol{\pi}',k).$$

Further, if $\boldsymbol{c} \neq \boldsymbol{c}'$, then

$$|\{i \mid c_i \neq c'_i \text{ or } \pi_i \neq \pi'_i\}| \geq |\{i \mid c_i \neq c'_i\}| \geq d$$

and similarly if $\boldsymbol{\pi} \neq \boldsymbol{\pi}'$. Hence,

$$d_H(\Lambda(\boldsymbol{c},\boldsymbol{\pi},k), \Lambda(\boldsymbol{c}',\boldsymbol{\pi}',k)) \geq qd. \tag{5}$$

In particular, this proves i).

To complete the proof, consider $k \neq k'$. Taking (4) modulo q we get

$$c_i \oplus_q k = c'_i \oplus_q k'$$

and so $c_i \neq c'_i$. Assume, without loss of generality, that $c_i < c'_i$. Let

$$j^* = q - 1 - c'_i.$$

Then $0 \leq j^* \leq q-2$. Let

$$s(j) = \{(c_i \oplus_q j) + k + \pi_i q\} - \{(c'_i \oplus_q j) + k' + \pi'_i q\}.$$

Since

$$c_i \oplus_q j^* = q - 1 - (c'_i - c_i), \qquad c'_i \oplus_q j^* = q - 1$$
$$c_i \oplus_q (j^*+1) = q - (c'_i - c_i), \qquad c'_i \oplus_q (j^*+1) = 0$$

we see that $s(j^*+1) - s(j^*) = q$. Hence,

$$s(j^*) \not\equiv s(j^*+1) \pmod{qn}$$

that is, (4) is satisfied for at most one of $j = j^*$ and $j = j^*+1$.

In particular, this shows that

$$\text{if } k \neq k', \quad \text{then } \Lambda(\boldsymbol{c},\boldsymbol{\pi},k) \neq \Lambda(\boldsymbol{c}',\boldsymbol{\pi}',k).$$

From this analysis, we see that when $k \neq k'$, then (4) can be satisfied only if $c_i \neq c'_i$, which is the case for at most $d_{\max}(C)$ values of i, and for each of these i for at most $q-1$ values of j; in total, for at most $(q-1)d_{\max}(C)$ elements. Hence,

$$d_H(\Lambda(\boldsymbol{c},\boldsymbol{\pi},k), \Lambda(\boldsymbol{c}',\boldsymbol{\pi}',k')) \geq qn - (q-1)d_{\max}(C) \geq n. \tag{6}$$

Combining (5) and (6), we get (3).

Suppose C and Π are balanced. Consider a fixed position $jq+i$ of the codewords of $\Lambda(C,\Pi)$. Then the element in this position in $\Lambda(\boldsymbol{c},\boldsymbol{\pi},0)$ is $(c_i \oplus_q j) + \pi_i q$. When $\boldsymbol{c}$ runs through the code C, then c_i and hence $(c_i \oplus_q j)$ runs through Z_q M/q times. Similarly, π_i runs through Z_n μ/n times (independently). Hence, $(c_i \oplus_q j) + \pi_i q$ runs through Z_{qn} $M\mu/(qn)$ times, that is, $\Lambda(C,\Pi)$ is balanced. Similarly, $\Lambda^*(C,\Pi)$ is balanced. QED

From Theorem 1, we immediately get the following corollary.

Corollary 1: For all n, q, d, and d' we have

i) $P(qn,qd) \geq A_q(n,d)P(n,d)$;

ii) if $qd \leq qn - (q-1)d'$, then $P(qn,qd) \geq qA_q(n,d,d')P(n,d)$;

iii) in particular, if $qd \leq n$, then $P(qn,qd) \geq qA_q(n,d)P(n,d)$;

iv) if $qd > qn - (q-1)d'$, then $P(qn, qn-(q-1)d') \geq qA_q(n,d,d')P(n,d)$.

Since $qd \leq n$ implies $q(qd) \leq qn$, we can use Corollary 1 part iii) repeatedly to get the following result.

Corollary 2: If $qd \leq n$, then

$$P(q^s n, q^s d) \geq q^s P(n,d) \prod_{i=0}^{s-1} A_q(q^i n, q^i d) \tag{7}$$

for all $s \geq 0$.

III. Examples

Example 2: Let $C = \{000, 101, 011, 110\}$, the binary even-weight code, and let $\Pi = S_3$. C is a $(3,4,2)$ code and Π is a $(3,6,2)$ PA.

By Theorem 1 part i), $\Lambda(C,\Pi)$ is a $(6,24,4)$ PA. The permutations in $\Lambda(C,\Pi)$ are the following:

$$\begin{array}{llllll} 024135, & 042153, & 204315, & 240351, & 402513, & 420531, \\ 125034, & 143052, & 305214, & 341250, & 503412, & 521430, \\ 035124, & 053142, & 215304, & 251340, & 413502, & 431520, \\ 134025, & 152043, & 314205, & 350241, & 512403, & 530421. \end{array}$$

It is easy to check that the distances between these permutations all are 4 or 6, that is, this PA is bidistant. Bidistant permutation arrays were studied in [11]. It is an open question if 24 is the maximal size of a PA of length 6 and distances 4 and 6.

Since $d_{\max}(C) = 2$, $2n - d_{\max}(C) = 4$. By Theorem 1 part ii), $\Lambda^*(C,\Pi)$ is a $(6,48,4)$ PA. The permutations in $\Lambda^*(C,\Pi) \setminus \Lambda(C,\Pi)$ are the following:

$$\begin{array}{llllll} 135240, & 153204, & 315420, & 351402, & 513024, & 531042, \\ 230145, & 254103, & 410325, & 452301, & 014523, & 032541, \\ 140235, & 104253, & 320415, & 302451, & 524013, & 542031, \\ 245130, & 203154, & 425310, & 401352, & 023514, & 041532. \end{array}$$

Since both C and Π are balanced, $\Lambda(C,\Pi)$ and $\Lambda^*(C,\Pi)$ are both balanced.

Example 3: Example 2 can be generalized. Let $q \geq 2$, $n \geq 2$, $\Pi = S_n$, and let C be the q-ary $(n, q^{n-1}, 2; q)$ code consisting of all $(c_0, c_1, \ldots, c_{n-1}) \in Z_q^n$ such that

$$\sum_{i=0}^{n-1} c_i \equiv 0 \pmod{q}.$$

By Theorem 1, $\Lambda(C,\Pi)$ is a $(qn, q^{n-1}n!, 2q)$ PA, and $\Lambda^*(C,\Pi)$ is a $(qn, q^n n!, 2q)$ PA for $n \geq 2q$. In particular

$$P(nq, 2q) \geq q^n n! \text{ for } n \geq 2q.$$

Both $\Lambda(C,\Pi)$ and $\Lambda^*(C,\Pi)$ are balanced.

Example 4: Let q be a prime power and $n = (q^m-1)/(q-1)$. Let C be the $(n, q^{n-m}, 3; q)$ Hamming code. It is known that $P(n,3) = n!/2$ (see [8, Theorem 1]) and an $(n, n!/2, 3)$ PA is the alternating group, let this be Π. By Theorem 1, $\Lambda(C,\Pi)$ is a $(qn, q^{n-m}n!/2, 3q)$ PA, and $\Lambda^*(C,\Pi)$ is a $(qn, q^{n-m+1}n!/2, 3q)$ PA for $n \geq 3q$. In particular

$$P(nq, 3q) \geq q^{n-m+1} n!/2, \qquad \text{for } n = \frac{q^m-1}{q-1} \geq 3q.$$

Both $\Lambda(C,\Pi)$ and $\Lambda^*(C,\Pi)$ are balanced.

Example 5: As a simple example of Corollary 1 part i), we consider $n = 12$ and $d = 5$. Chu *et al.* [3] showed that $P(12, 5) \geq 3 \cdot 243760$. Since $A_3(12, 5) \geq 2^6$ (see, e.g., [13]), we get

$$P(36, 15) \geq 3^7 \cdot 243760.$$

Further, Greferath and Schmidt [9] showed that $A_3(36, 15) \geq 3^{12}$, and hence we also get

$$P(108, 45) \geq 3^{19} \cdot 243760.$$

We now consider some special cases of Corollary 2.

Example 6: If p is a prime and $j \geq 1$, then there exists a generalized first-order Reed–Muller $\left(p^j, p^{\binom{j+p-1}{p-1}}, p^{j-1}; p\right)$ code (see [1, p. 1300]). In particular, for $i \geq 0$ we have

$$A_p(p^{i+2}, p^{i+1}) \geq p^{\binom{i+p+1}{p-1}}. \tag{8}$$

Hence, letting $q = p$, $n = p^2$, $d = p$, and $s = m - 2$ in (7), we get

$$P(p^m, p^{m-1}) \geq p^{m-2+\sum_{i=0}^{m-3}\binom{i+p+1}{p-1}} P(p^2, p). \tag{9}$$

Example 7: In general, we do not know the value of $P(p^2, p)$. However, $P(4, 2) = 24$ and S_4 is a $(4, 24, 2)$ PA. Hence, for $m \geq 2$, (9) gives

$$P(2^m, 2^{m-1}) \geq 24 \prod_{i=4}^{m+1} 2^i = 3 \cdot 2^{\frac{(m-1)(m+4)}{2}}. \tag{10}$$

Moreover, since both the first-order Reed–Muller code and S_4 are balanced, all the $(2^m, 3 \cdot 2^{\frac{(m-1)(m+4)}{2}}, 2^{m-1})$ PAs obtained are balanced.

To illustrate the construction of these PAs, let C_j be the $(2^j, 2^{j+1}, 2^{j-1})$ first-order Reed–Muller code and Π_m be the $(2^m, 2^{m-1})$ PA constructed. Then

$$\begin{aligned}
\Pi_2 &= S_4 \\
\Pi_3 &= \Lambda^*(C_2, \Pi_2) = \Lambda^*(C_2, S_4) \\
\Pi_4 &= \Lambda^*(C_3, \Pi_3) = \Lambda^*\left(C_3, \Lambda^*(C_2, S_4)\right) \\
\Pi_5 &= \Lambda^*(C_4, \Pi_4) = \Lambda^*\left(C_4, \Lambda^*(C_3, \Lambda^*(C_2, S_4))\right)
\end{aligned}$$

etc.

Wadayama and Vinck (see [17, Corollary 1]) presented a multilevel construction for permutation arrays. Using the first-order Reed–Muller codes in the multilevel construction, they obtained a $(2^m, 2^{m-1})$ PA of size

$$12 \prod_{i=4}^{m+1} (2^i - 2).$$

For $m = 3$, 4, and 5, this gives $(8, 168, 4)$, $(16, 5040, 8)$, and $(32, 312480, 16)$ PAs, respectively. We note that (10) gives substantial improvements over these results for all m. For example, for $m = 3$, 4, and 5 we get $(8, 384, 4)$, $(16, 12288, 8)$, and $(32, 786432, 16)$ PAs, respectively. We note that Chu *et al.* [3] recently found a $(8, 2688, 4)$ PA by computer search.

Example 8: As noted in Example 4, $P(n, 3) = n!/2$ for all n, and so

$$P(9, 3) = 9!/2 = 2240 \cdot 3^4.$$

Hence, (9) gives

$$P(3^m, 3^{m-1}) \geq 2240 \cdot 3^{(m-1)(m^2+4m+12)/6}. \tag{11}$$

Again, all the PAs obtained are balanced. The bound (11) is nice because it is explicit. However, since the proof of (9) was based on the bound (8) which usually can be improved when $p > 2$, the bound (11) can also be improved. For example, using the bounds on $A_3(3^j, 3^{j-1})$ implied by the best known linear codes (see the tables in [1, pp. 371–418]) we get the following bounds:

m	2	3	4	5	6
$P(3^m, 3^{m-1})/2240 \geq$	3^4	3^{11}	3^{25}	3^{49}	3^{106}

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REFERENCES

[1] V. S. Pless and W. C. Huffman, Eds., *Handbook of Coding Theory*. Amsterdam, The Netherlands: Elsevier, 1998.

[2] I. Blake, G. Cohen, and M. Deza, "Coding with permutations," *Inform. Contr.*, vol. 43, no. 1, pp. 1–19, 1979.

[3] W. Chu, C. J. Colbourn, and P. Dukes, "Constructions for permutation arrays in powerline communications," manuscript, submitted for publication.

[4] C. J. Colbourn, T. Kløve, and A. C. H. Ling, "Permutation arrays for powerline communication and mutually orthogonal latin squares," *IEEE Trans. Inform. Theory*, to be published.

[5] M. Deza and S. A. Vanstone, "Bounds on permutation arrays," *J. Statist. Planning and Inference*, vol. 2, pp. 197–209, 1978.

[6] C. Ding, F.-W. Fu, T. Kløve, and V. K. Wei, "Constructions of permutation arrays," *IEEE Trans. Inform. Theory*, vol. 48, pp. 977–980, Apr. 2002.

[7] H. C. Ferreira and A. J. H. Vinck, "Inference cancellation with permutation trellis arrays," in *Proc. IEEE Vehicular Technology Conf.*, 2000, pp. 2401–2407.

[8] P. Frankl and M. Deza, "On the maximum number of permutations with given maximal and minimal distance," *J. Comb. Theory*, ser. A, vol. 22, pp. 352–360, 1977.

[9] M. Greferath and S. E. Schmidt, "Gray isometries for finite chain rings and a nonlinear ternary $(36, 3^{12}, 15)$ code," *IEEE Trans. Inform. Theory*, vol. 45, pp. 2522–2524, Nov. 1999.

[10] T. Kløve, "Classification of permutation codes of length 6 and minimum distance 5," in *Proc. Int. Symp. Information Theory and Its Applications*, 2000, pp. 465–468.

[11] R. Mathon and A. P. Street, "Overlarge sets of 2- (11,5,2) designs and related configurations," *Discr. Math.*, vol. 255, pp. 275–286, 2002.

[12] H. Tarnanen, "Upper bounds on permutation codes via linear programming," *Europ. J. Combin.*, vol. 20, pp. 101–114, 1999.

[13] R. J. M. Vaessens, E. H. L. Aarts, and J. H. van Lint, "Genetic algorithms in coding theory—A table for $A_3(n, d)$," *Discr. Appl. Math.*, vol. 45, pp. 71–87, 1993.

[14] A. J. H. Vinck, "Coded modulation for powerline communications," *AEÜ Int. J. Electron. Commun.*, vol. 54, no. 1, pp. 45–49, 2000.

[15] A. J. H. Vinck and J. Häring, "Coding and modulation for power-line communications," in *Proc. Int. Symp. Power Line Communication*, Limerick, Ireland, Apr. 5–7, 2000.

[16] A. J. H. Vinck, J. Häring, and T. Wadayama, "Coded M-FSK for power-line communications," in *Proc. IEEE Int. Symp. Information Theory*, Sorrento, Italy, 2000, p. 137.

[17] T. Wadayama and A. J. H. Vinck, "A multilevel construction of permutation codes," *IEICE Trans. Fundamentals Electron., Commun. Comp. Sci.*, vol. 84, pp. 2518–2522, 2001.