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Partial permutation decoding for codes from affine geometry designs

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Abstract. We find explicit PD-sets for partial permutation decoding of the generalized Reed-Muller codes $\mathcal{R}_{\mathbb{F}_p}(2(p-1), 3)$ from the affine geometry designs $AG_{3,1}(\mathbb{F}_p)$ of points and lines in dimension 3 over the prime field of order p, using the information sets found in [8].

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1. Introduction

In [7] we found s-PD-sets (see Definition 1) for $s = 2$ and 3 for partial permutation decoding for the p-ary codes of affine planes of prime order p; this was extended to projective planes. Since PD-sets are dependent on specific information sets for the codes, we were able to deal with the plane case by using information sets deduced from the bases found by Moorhouse [12]. Using new information sets found in [8], we extended these results to the codes from the designs of points and hyperplanes of affine and projective geometries of prime order, obtaining 2-PD-sets. We now use these information sets to find s-PD-sets for $s = 2$ and 3 for the p-ary codes of the affine geometry designs $AG_{3,1}(\mathbb{F}_p)$ of points and lines in 3-dimensional affine space $AG_3(\mathbb{F}_p)$ over the field $\mathbb{F}_p$. We prove the following theorem:

THEOREM 1. *Let $\mathcal{D}$ be the 2-$(p^3, p, 1)$ design $AG_{3,1}(\mathbb{F}_p)$ of points and lines in the affine space $AG_3(\mathbb{F}_p)$, where p is a prime, and let $C = \mathcal{R}_{\mathbb{F}_p}(2(p-1), 3)$ be the p-ary code of $\mathcal{D}$. Then C is a $[p^3, \frac{1}{6}p(5p^2+1), p]_p$ code with information set*

$$\mathcal{I} = \{(i_1, i_2, i_3) \mid i_k \in \mathbb{F}_p,\ 1 \le k \le 3,\ \sum_{k=1}^{3} i_k \le 2(p-1)\}. \tag{1}$$

Let T be the translation group of $AG_3(\mathbb{F}_p)$, let D be the group of invertible diagonal 3×3 matrices, and let Z be the group of scalar matrices. For each $d \in \mathbb{F}_p$ with $d \neq 0$, let $\mu(d)$ be the associated dilatation. Corresponding to the information set $\mathcal{I}$, the code C has a

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2-PD-set of the form $T \cup T\mu(d)$ of size $2p^3$ for $p \geq 5$ and for some $d \in \mathbb{F}_p^*$, and the group TD is a 3-PD-set for C of size $p^3(p-1)^3$ for $p \geq 7$. (In fact, for the 2-PD-set, we can choose $d = (p-1)/2$.)

It should be noted that, when elements of $\mathbb{F}_p$ occur in an inequality, they are being treated as integers in the interval $[0, p-1]$.

The proof of the theorem will follow in Section 3, after a section on some basic results, definitions and background. In Section 4 we obtain a new 3-PD-set for the p-ary code $AG_{2,1}(\mathbb{F}_p)$ of points and lines in the affine plane $AG_2(\mathbb{F}_p)$ over the field $\mathbb{F}_p$.

2. Background

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) *design*, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The *code* $C_p(\mathcal{D})$ of $\mathcal{D}$ over the finite field $\mathbb{F}_p$, is the space spanned by the incidence vectors of the blocks over $\mathbb{F}_p$, and is thus a subspace of $\mathbb{F}_p^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to $\mathbb{F}_p$.

The notation $[n, k, d]_q$ will denote a linear code C of length n, dimension k, and minimum weight d, over the field $\mathbb{F}_q$. A *generator matrix* for the code is a $k \times n$ matrix made up of a basis for C. The *dual* code $C^{\perp}$ is the orthogonal subspace under the standard inner product $(,)$, i.e. $C^{\perp} = \{v \in \mathbb{F}_q^n | (v, c) = 0 \text{ for all } c \in C\}$. A *check* matrix for C is a generator matrix H for $C^{\perp}$; the *syndrome* of a vector $y \in \mathbb{F}_q^n$ is Hy^T. Two linear codes of the same length and over the same field are *isomorphic* if they can be obtained from one another by permuting the coordinate positions. (See Huffman [6] for related, more general, concepts of isomorphisms of codes.) Any linear code is isomorphic to a code with generator matrix in so-called *standard form*, i.e. the form $[I_k \mid A]$; a check matrix then is given by $[-A^T \mid I_{n-k}]$. The first k coordinates are the *information symbols* (or set) and denoted by $\mathcal{I}$, and the last $n-k$ coordinates are the *check symbols*, denoted by $\mathcal{C}$. An *automorphism* of a code C is an isomorphism from C to C. The automorphism group will be denoted by $\mathrm{Aut}(C)$.

For any finite field $\mathbb{F}_q$ of order q, the set of points and r-dimensional subspaces of an m-dimensional projective geometry forms a 2-design which we will denote by $PG_{m,r}(\mathbb{F}_q)$. Similarly, the set of points and r-dimensional flats of an m-dimensional affine geometry forms a 2-design, $AG_{m,r}(\mathbb{F}_q)$. The *automorphism groups* of these designs (and codes) are the full projective or affine semi-linear groups, $P\Gamma L_{m+1}(\mathbb{F}_q)$ or $A\Gamma L_m(\mathbb{F}_q)$, and are always 2-transitive on points. If $q = p^e$ where p is a prime, the codes of these designs are over $\mathbb{F}_p$ and are subfield subcodes of the generalized Reed-Muller codes: see [1, Chapter 5] for a full treatment. The dimension and minimum weight is known in each case: see [1, Theorem 5.7.9].

Permutation decoding was first developed by MacWilliams [10] and involves finding a set of automorphisms of a code called a PD-set. The method is described fully in MacWilliams and Sloane [11, Chapter 15] and Huffman [6, Section 8]. We extend the concept of PD-sets to s-PD-sets for s-error-correction in [7], as in the following definition. This coincides with the use of the term s-PD-set in Kroll and Vincenti [9].

DEFINITION 1. If C is a t-error-correcting code with information set $\mathcal{I}$ and check set $\mathcal{C}$, then a *PD-set* for C is a set $\mathcal{S}$ of automorphisms of C which is such that every t-set of coordinate positions is moved by at least one member of $\mathcal{S}$ into $\mathcal{C}$.

For $s \leq t$ an *s-PD-set* is a set $\mathcal{S}$ of automorphisms of C which is such that every s-set of coordinate positions is moved by at least one member of $\mathcal{S}$ into $\mathcal{C}$.

That a PD-set will fully use the error-correction potential of the code follows easily and is proved in Huffman [6, Theorem 8.1], and that an s-PD-set will correct s errors follows in a similar manner. The algorithm for permutation decoding is given in [6, 11] or see [7]. Such sets might not exist at all, and the property of having a PD-set will not, in general, be invariant under isomorphism of codes, i.e. it depends on the choice of $\mathcal{I}$ and $\mathcal{C}$. Furthermore, there is a bound on the minimum size of $\mathcal{S}$ (see [5], [13], or [6]). This bound can be adapted to one for s-PD-sets by replacing in the formula for the bound, the variable t, that denotes full error-correction, by $s < t$ for correction of s errors.

To obtain PD-sets, a generator matrix for the code needs to be in standard form, and thus the question of what points to take as information symbols arises.

We use the notation of [1, Chapter 5] or [2] for generalized Reed-Muller codes: (see [1, Definition 5.4.1]):

DEFINITION 2. Let $V = \mathbb{F}_q^m$ be the vector space of m-tuples, for $m \geq 1$, over $\mathbb{F}_q$, where $q = p^t$ and p is a prime. For any ρ such that $0 \leq \rho \leq m(q-1)$, the *ρ^{th}-order generalized Reed-Muller code* $\mathcal{R}_{\mathbb{F}_q}(\rho, m)$ is the subspace of $\mathbb{F}_q^V$ (with basis the characteristic functions of vectors in V) of all m-variable polynomial functions (reduced modulo $x_i^q - x_i$) of degree at most ρ. Thus

$$\mathcal{R}_{\mathbb{F}_q}(\rho, m) = \langle x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \mid 0 \leq i_k \leq q-1, \, for 1 \leq k \leq m, \, \sum_{k=1}^{m} i_k \leq \rho \rangle.$$

These codes are thus codes of length q^m and the codewords are obtained by evaluating the m-variable polynomials in the subspace at all the points of the vector space $V = \mathbb{F}_q^m$.

The code $\mathcal{R}_{\mathbb{F}_p}((m-1)(p-1), m)$ is the p-ary code of the affine geometry design $AG_{m,1}(\mathbb{F}_p)$ of points and lines in affine space $AG_m(\mathbb{F}_p)$: see [1, Theorem 5.7.9]. Here we take $m = 3$, in which case $\mathcal{R}_{\mathbb{F}_p}(2(p-1), 3)$ is a $[p^3, \frac{1}{6}p(5p^2+1), p]_p$ code over $\mathbb{F}_p$.

The information set we will be using was found in [8, Theorem 1, Corollary 2]:

RESULT 1. *If p is a prime, the code $\mathcal{R}_{\mathbb{F}_p}(\nu, m)$ has information set*

$$\mathcal{I} = \{(i_1, \dots, i_m) \mid i_k \in \mathbb{F}_p,\ 1 \leq k \leq m,\ \sum_{k=1}^{m} i_k \leq \nu\}. \tag{2}$$

3. Proof of theorem

Before proving the theorem, we establish some notation. We will use τ with an appropriate argument to denote translations in $\mathbb{F}_p$ and $AG_3(\mathbb{F}_p)$. Thus, $\tau(w) : v \mapsto v + w$. If $w = (w_1, w_2, w_3)$, where $w_1, w_2, w_3 \in \mathbb{F}_p$, we will also write $\tau(w)$ as $\tau(w_1, w_2, w_3)$. For $d_1, d_2, d_3 \in \mathbb{F}_p \backslash \{0\}$, let $\delta(d_1)$ denote the mapping $v_1 \mapsto d_1 v_1$, for $v_1 \in \mathbb{F}_p$ and let $\delta(d_1, d_2, d_3)$ denote the mapping $(v_1, v_2, v_3) \mapsto (d_1 v_1, d_2 v_2, d_3 v_3)$, for $v_1, v_2, v_3 \in \mathbb{F}_p$.

We begin the proof of Theorem 1 by establishing that there is a 2-PD-set of the stated form. Let $\mathcal{C}$ denote the check set of C corresponding to the information set $\mathcal{I}$, where

$$\mathcal{I} = \{(i_1, i_2, i_3) \mid i_k \in \mathbb{F}_p,\ 1 \leq k \leq 3,\ \sum_{k=1}^{3} i_k \leq 2(p-1)\}$$

as in Equation (1). Let P' and Q' be two points. By a translation τ', we can take Q' to $Q = (0, 0, 0)$ and P' to $P = (a, b, c)$.

If $a, b \leq (p-3)/2$, let $w = (p-1-a, p-1-b, e)$ where $e = p-1$ or $p-2$ according as $c \neq 1$ or $c = 1$. Clearly, $P\tau(w) = (p-1, p-1, c+e) \in \mathcal{C}$ as $c + e \neq 0$. Also, $p - 1 - a + p - 1 - b \geq p + 1$ and $e \geq p - 2$. So, $Q\tau(w) \in \mathcal{C}$.

If $a, b \geq (p+3)/2$, let $w = (p-1, p-1, e)$ where $e = p-1-c$ or $p-2-c$ according as $c \neq p-1$ or $c = p-1$. Then, $Q\tau(w) = (p-1, p-1, e) \in \mathcal{C}$ as $e \neq 0$. Since $P\tau(w) = (a-1, b-1, c+e)$ and $a + b - 2 \geq p + 1$ and $c + e \geq p - 2$, $P\tau(w) \in \mathcal{C}$.

If $a \leq (p-3)/2$, $b \geq (p-1)/2$, and $c = (p-1)/2$, let $w = (p-1-a, p-1, (p-1)/2)$. Clearly, $Q\tau(w) \in \mathcal{C}$. Also, $P\tau(w) = (p-1, b-1, p-1) \in \mathcal{C}$.

If $a \leq (p+1)/2$, $b \geq (p+3)/2$, and $c = (p+1)/2$, let $w = (p-1-a, p-1, p-1)$. Clearly, $Q\tau(w) \in \mathcal{C}$. Also, $P\tau(w) = (p-1, b-1, (p-1)/2)$. Since $b - 1 \geq (p+1)/2$, $P\tau(w) \in \mathcal{C}$.

If $a \geq (p+5)/2$ and $b = c = (p-1)/2$ let $w = (p-1, p-1, p+2-a)$. Clearly, $Q\tau(w) \in \mathcal{C}$. Also, $P\tau(w) = (a-1, (p-3)/2, 3(p+1)/2 - a) \in \mathcal{C}$.

If $a \leq (p-5)/2$ and $b = c = (p+1)/2$ let $w = ((p+3)/2, (p-3)/2, p-1)$. Clearly, $Q\tau(w) \in \mathcal{C}$. Also, $P\tau(w) = (a+(p+3)/2, p-1, (p-1)/2)$. Since $(p+3)/2 \leq a \leq p-1$, $P\tau(w) \in \mathcal{C}$.

These arguments can be applied to any permutation of the coordinates. So, in these cases, we can find a translation τ'' so that $P'\tau'\tau''$, $Q'\tau'\tau'' \in \mathcal{C}$. Hence, the only cases that remain are when at least two of a, b and c are in $\{(p-1)/2, (p+1)/2\}$ and, if there is a remaining one, it is in $\{(p-3)/2, (p+3)/2\}$.

If $p > 7$, then none of $2a$, $2b$ and $2c$ are in $\{(p-3)/2, (p-1)/2, (p+1)/2, (p+3)/2\}$. The preceding arguments show the existence of a translation τ'' for which $P'\tau'\delta(2)\tau''$ and $Q'\tau'\delta(2)\tau''$ are in $\mathcal{C}$. If $p = 5$ or $p = 7$, we can apply the same argument to $a(p-1)/2$, $b(p-1)/2$, and $c(p-1)/2$, even though the sets $\{a(p-1)/2, b(p-1)/2, c(p-1)/2\}$ and $\{(p-3)/2, (p-1)/2, (p+1)/2, (p+3)/2\}$ overlap. Hence, in these cases, there is a translation τ'' for which $P'\tau'\delta((p-1)/2)\tau''$, $Q'\tau'\delta((p-1)/2)\tau'' \in \mathcal{C}$.

Since the translations form a normal subgroup of the automorphism group of $AG_3(\mathbb{F}_p)$, we can write $\tau'\delta(d)\tau'' = \tau\delta(d)$, for some translation τ. Hence, we have shown that $T \cup T\delta(d)$ is a 2-PD-set for C with d chosen as in the preceding paragraph. In fact, we could take $d = (p-1)/2$ in all cases; the details are straightforward but would lengthen the proof. This completes the proof of the first part of the theorem.

Next, we show that TD, the group generated by T and D, where $D = \{\delta(d_1, d_2, d_3) \mid d_1, d_2, d_3 \in \mathbb{F}_p\backslash\{0\}\}$, is a 3-PD-set for C.

A translation can take any three points to the triple $X = (0, 0, 0)$, $P = (a, b, c)$, $Q = (d, e, f)$ where not all of a, b, c, d, e, f are 0 and $(a, b, c) \neq (d, e, f)$. A point (a, b, c) is in the check set $\mathcal{C}$ if, and only if, $a + b + c \geq 2p - 1$. The theme of the proof is to show that, by a non-zero multiplication and an addition on each coordinate position, the three entries (either $[0, a, d]$, $[0, b, e]$ or $[0, c, f]$) in that position can be moved to three elements of $\mathbb{F}_p$ corresponding to integers in the interval $[(2p-1)/3, p-1]$. If, in the i-th coordinate position, the multiplication is by d_i and the addition is w_i, then this mapping as effected by an element $\delta(d_1, d_2, d_3)\tau(w_1, w_2, w_3)$ of DT $(= TD)$ necessarily maps the triple X, P and Q into $\mathcal{C}$.

This approach needs to be modified for $p = 13$ and fails to work for $p = 7$. In the case $p = 7$, we have checked the result with simple computer programs using Magma [3] and GAP [4].

We deal first with some easy cases. If all three entries are 0, then $\tau(p-1)$ has the desired effect; that is, $\tau(p-1)$ acting on the entries maps $[0, 0, 0]$ to $[p-1, p-1, p-1]$. If two entries are 0 and one is nonzero, say $[0, 0, d]$, then $\delta(d^{-1})\tau(p-2)$ has the desired effect. Thus, we need only consider triples with one 0 and two nonzero elements. These may be mapped, by a suitable nonzero multiplication, to $[0, 1, g]$, where $1 \leq g \leq p-1$.

We now subdivide the proof into two cases, according as $p \equiv 1 \pmod 6$ or $p \equiv 5 \pmod 6$. We write $p = 6m + 1$ in the former case and $p = 6m + 5$ in the latter. Note that $m \geq 1$ in both cases, since $p \geq 7$.

Case 1: $p = 6m + 1$. In this case, $(2p - 1)/3 < 4m + 1$. Since we do not consider $p = 7$ here, $m \geq 2$.

If $1 \leq g \leq 2m - 1$, $[0, 1, g]\tau(4m + 1) = [4m + 1, 4m + 2, 4m + 1 + g]$ and $4m + 1 < 4m + 1 + g \leq 6m$. If $4m + 3 \leq g \leq 6m$, $[0, 1, g]\tau(6m - 1) = [6m - 1, 6m, g - 2]$ and $4m + 1 \leq g - 2 \leq 6m - 2$.

If $2m + 2 \leq g \leq 3m$, $[0, 1, g]\delta(2)\tau(6m - 2) = [6m - 2, 6m, 2g - 3]$ and $4m + 1 \leq 2g - 3 \leq 6m - 3$. If $3m + 1 \leq g \leq 4m$, $[0, 1, g]\delta(2)\tau(4m + 1) = [4m + 1, 4m + 3, 2g - 2m]$ and $4m + 2 \leq 2g - 2m \leq 6m$.

This leaves just four values of g to consider, *viz.* $g = 2m, 2m + 1, 4m + 1, 4m + 2$. Noting that $4m + 4 \leq 6m$, for $g = 2m + 1$, $[0, 1, g]\delta(3)\tau(4m + 1) = [4m + 1, 4m + 4, 4m + 3]$ and for $g = 4m + 1$, $[0, 1, g]\delta(3)\tau(4m + 1) = [4m + 1, 4m + 4, 4m + 2]$. For the other two values of g, we require $6m - 4 \geq 4m + 1$; that is, $m \geq 3$, i.e. $p \geq 19$. If $g = 2m$, $[0, 1, g]\delta(3)\tau(6m - 3) = [6m - 3, 6m, 6m - 4]$. If $g = 4m + 2$, $[0, 1, g]\delta(3)\tau(6m - 4) = [6m - 4, 6m - 1, 6m]$.

We now deal with the last two values of g when $p = 13$ $(m = 2)$. For $g = 4$, note that $[0, 1, 4]\tau(8) = [8, 9, 12]$, $[0, 1, 4]\delta(9)\tau(12) = [12, 8, 9]$ and $[0, 1, 4]\delta(3)\tau(8) = [9, 12, 8]$. For any coordinate column of this type, we can choose a mapping in which one of the entries is 8 $(= 4m)$ while the others are $\geq 4m + 1$. Moreover, the $4m$ entry can be made to appear in the image of any one of our triple of points X, P and Q. Similarly, for $g = 10$, $[0, 1, 10]\delta(3)\tau(8) = [8, 11, 12]$, $[0, 1, 10]\delta(12)\tau(9) = [9, 8, 12]$ and $[0, 1, 10]\tau(11) = [11, 12, 8]$.

We can thus arrange that the image of each of the points X, P and Q has at most one entry equal to $4m$ while the others are $\geq 4m + 1$. Hence, these images lie in $\mathcal{C}$. This completes the proof of Case 1.

Case 2: $p = 6m + 5$. In this case, $(2p - 1)/3 = 4m + 3$ and $m \geq 1$.

If $1 \leq g \leq 2m + 1$, $[0, 1, g]\tau(4m + 3) = [4m + 3, 4m + 4, 4m + 3 + g]$ and $4m + 3 < 4m + 3 + g \leq 6m + 4$. If $4m + 5 \leq g \leq 6m + 4$, $[0, 1, g]\tau(6m + 3) = [6m + 3, 6m + 4, g - 2]$ and $4m + 3 \leq g - 2 \leq 6m + 2$.

If $2m + 3 \leq g \leq 3m + 2$, $[0, 1, g]\delta(2)\tau(6m + 2) = [6m + 2, 6m + 4, 2g - 3]$ and $4m + 3 \leq 2g - 3 \leq 6m + 1$. If $3m + 3 \leq g \leq 4m + 3$, $[0, 1, g]\delta(2)\tau(4m + 3) = [4m + 3, 4m + 5, 2g - 2m - 2]$ and $4m + 4 \leq 2g - 2m - 2 \leq 6m + 4$.

This leaves just two values of g to consider. If $g = 2m + 2$, $[0, 1, g]\delta(3)\tau(4m + 3) = [4m + 3, 4m + 6, 4m + 4]$. If $g = 4m + 4$, $[0, 1, g]\delta(3)\tau(4m + 3) = [4m + 3, 4m + 6, 4m + 5]$. This completes the proof of Case 2 and the proof of the theorem. □

We illustrate the method of proof for the 3-PD-sets with an example for $p = 19 = 6m + 1$ where $m = 3$ and $4m + 1 = 13$. Suppose our three points have been mapped by a translation

τ' to the points $(0, 0, 0)$, $(2, 11, 5)$, $(3, 10, 7)$. For the first coordinate triple $[0, 2, 3]$, the map $\delta(10)$ takes this to the standard form $[0, 1, 11]$ and the map $\delta(2)\tau(13)$ takes this to the triple $[13, 15, 16]$. For the second coordinate triple $[0, 11, 10]$, the map $\delta(7)$ takes it to $[0, 1, 13]$ and the map $\delta(3)\tau(13)$ takes this to the triple $[13, 16, 14]$. For the third coordinate triple $[0, 5, 7]$, the map $\delta(4)$ takes this to $[0, 1, 9]$ and the map $\delta(2)\tau(16)$ takes this to the triple $[16, 18, 15]$. Note that $\delta(10)\delta(2) = \delta(1)$, $\delta(7)\delta(3) = \delta(2)$ and $\delta(4)\delta(2) = \delta(8)$. Thus, the element $\tau'\delta(1, 2, 8)\tau(13, 13, 16)$ of TD will take our original three points to the points $(13, 13, 16)$, $(15, 16, 18)$, $(16, 14, 15)$, all of which are in the check set $\mathcal{C}$.

Note: These codes have high rate $\geq .83$. The worst-case time-complexity for the decoding algorithm using an s-PD-set of size z on a code of length n and dimension k is $\mathcal{O}(nkz)$, as a simple counting argument shows.

4. Affine planes

In [7, Proposition 4.5] we found 3-PD-sets of size $2p^2(p-1)$ for the codes from the affine planes $AG_{2,1}(\mathbb{F}_p)$, using an information set different from the one we have used in Theorem 1. We show that this can be improved to $p^2(p-1)$ using the set $\mathcal{I}$ of Equation 1. This further leads to $(m+1)$-PD-sets for the codes of the designs $AG_{m,m-1}(\mathbb{F}_p)$, using [8, Proposition 4]

PROPOSITION 1. *Let p be a prime. Let $\mathcal{D}$ be the design $AG_{2,1}(\mathbb{F}_p)$ of points and lines in the affine plane $AG_2(\mathbb{F}_p)$ and let $C = \mathcal{R}_{\mathbb{F}_p}(p-1, 2)$ be the p-ary code of $\mathcal{D}$. With information set*

$$\mathcal{I} = \{(i_1, i_2) \mid i_k \in \mathbb{F}_p,\ 1 \leq k \leq 2,\ \sum_{k=1}^{2} i_k \leq p-1\},$$

the group TZ, where T is the translation group and Z is the group of scalar matrices, is a 3-PD-set for C for $p \geq 7$, of size $p^2(p-1)$.

Proof. We extend our notation τ and μ for translations and dilatations, as used in Theorem 1, to affine planes. Thus $Z = \{\mu(a) \mid a \in \mathbb{F}_p,\ a \neq 0\}$. Let $H = TZ$.

Any three distinct points may be mapped by a translation to a triple of the form $X = (0, 0)$, $P = (q, r)$, $Q = (s, t)$ where $(q, r) \neq (0, 0)$, $(s, t) \neq (0, 0)$ and $(q, r) \neq (s, t)$; in particular, $q \neq s$ or $r \neq t$. We may assume that $q \neq s$. The case $r \neq t$ may be dealt with in a similar manner. We will show how to find maps in TZ that move such triples into the check set $\mathcal{C}$.

Since $q \neq s$, some element of Z will fix X and map P and Q into a pair P' and Q' of the form (a, b), $(a+1, d)$, for some a, b, d, where $0 \leq a \leq p-2$. If $a \geq (p+1)/2$, $\mu(p-1)$

will fix X and map (a, b) to $(p-a, p-b)$ and $(a+1, d)$ to $(p-a-1, p-d)$; that is, to a similar triple with $a \le (p-3)/2$. Hence, we may assume that $a \le (p-1)/2$.

In this case, $p-a-2 \ge (p-3)/2$. The mapping $\tau(p-a-2, u)$ maps X, P' and Q' to $(p-a-2, u)$, $(p-2, u+b)$ and $(p-1, u+d)$, which are in $\mathcal{C}$ if $a+2 \le u \le p-1$ and $u \notin \{p-b, p-b+1, p-d\}$. Since $a+2 \le (p+3)/2$, there are at least $(p-3)/2$ integers in the interval $[a+2, p-1]$ of which at most 3 must be excluded. If $p \ge 11$, there is at least one value of u meeting these constraints.

The only case that remains is $p = 7$. We can apply the argument of the preceding paragraph if $a = 0$ or $a = 1$. We are left with $a = 2$ and $a = 3$.

The triple X, P' and Q' is mapped by $\tau(5-a, 6)$ into $\mathcal{C}$ if $b \neq 1$ or 2 and $d \neq 1$. If $d = 1$, $\tau(5-a, 5)$ or $\tau(6, 4)$ maps the triple into $\mathcal{C}$ according as $b \neq 2$ or $b = 2$. If $b = 1$, $\tau(5-a, 5)$, $\tau(3, 4)$ or $\tau(6, 4)$ maps the triple into $\mathcal{C}$ according as $d \neq 2$, $d = 2$ and $a = 2$ or $d = 2$ and $a = 3$. If $b = 2$ and $a = 2$, $\tau(3, 4)$ or $\mu(6)\tau(1, 6)$ maps the triple into $\mathcal{C}$ according as $d \neq 3$ or $d = 3$. If $b = 2$ and $a = 3$, $\mu(3)\tau(1, 6)$ or $\mu(3)\tau(3, 5)$ maps the triple into $\mathcal{C}$ according as $d \neq 5$ or $d = 5$.

This completes the proof of the proposition. □

Note: 1. We exclude $p = 5$ since the code is only 2-error-correcting.
2. Using [8, Proposition 4], we can now construct $(m+1)$-PD-sets of size $p^m(p-1)$ for $AG_{m,m-1}(\mathbb{F}_p)$, the design of points and hyperplanes in $AG_m(\mathbb{F}_p)$, for $m \ge 2$, p prime.

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