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PERFECT CODES IN THE LEE METRIC AND THE PACKING OF POLYOMINOES*

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1. The geometry of Shannon's five-phase code. In [4] Shannon considered the problem of coding to completely eliminate errors in a channel using a 5-symbol alphabet, with the error pattern as shown in Fig. 1. The alphabet may be regarded as the integers modulo 5. When the integer r is sent, either r or $r + 1$ is received, with respective probabilities p and q. If one forms a "code" consisting of sending each symbol m times to represent the fact that it occurred once in the message, then there is still a probability of q^m that an error will occur. However, there exists a code using only two code symbols per message symbol which eliminates errors entirely (see Fig. 2). In this code, if (a, b) is a codeword, then it may be received as either (a, b) or $(a + 1, b)$ or $(a, b + 1)$ or $(a + 1, b + 1)$. However, we can associate all four of these received messages *uniquely* with (a, b) when we use the code of Fig. 2. This is most readily seen via the geometric presentation in Fig. 3. The 25 possible codewords (a, b) are represented by the 25 cells, with coordinates (a, b). The codewords of Fig. 2 correspond to the cells with dots in them. Each dot is in the lower left-hand corner of its "ambiguity square." (The entire 5×5 array is to be regarded as a torus.) Since these ambiguity squares are nonoverlapping, any received message can be uniquely interpreted.

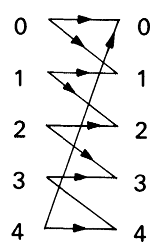

FIG. 1. *Shannon's 5-phase channel*

0 = (0, 0)

1 = (1, 2)

2 = (2, 4)

3 = (3, 1)

4 = (4, 3)

FIG. 2. *An error eliminating code for the channel in Fig.* 1

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The packing of five 2×2 squares into the 5×5 torus shown in Fig. 3 is reasonably efficient, but the resulting code is not close-packed. In particular, there are 5 unused cells in Fig. 3. For the channel described by the error statistics of Fig. 1, no further improvement is possible. However, if other errors are remotely possible, then it is advantageous to assign the 5 unused squares to the ambiguity regions of the 5 codewords. This can be done as in Fig. 4, where the error which occurs when (a, b) is received as $(a - 1, b)$ will also be corrected. Since there are no open spaces in Fig. 4, this code is "close-packed" and corresponds geometrically to a tiling of the 5×5 torus with P-pentominoes.

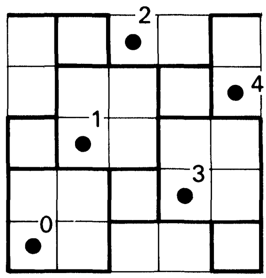

FIG. 3. *Geometric representation of the code in Fig.* 2

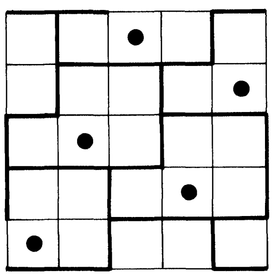

FIG. 4. *A close-packed P-pentomino code*

In general, any tiling of an $n \times n$ torus by translations of a given polyomino shape corresponds to a close-packed code, using word length 2 over the n symbol alphabet. However, the error patterns corrected by such a code are likely to be unnatural or infrequent ones, unless the shape of the polyomino is constrained in various ways. We shall next consider a class of polyominoes which satisfy the appropriate constraints.

2. Two-dimensional codes in the Lee metric. In Fig. 5, we see the polyomino generated by taking the codeword (a, b) and displacing either component by 1 unit, either up or down. The resulting figure, an X-pentomino, is accordingly a "sphere of radius one" with center at (a, b) in the metric (called the Lee metric) which computes the sum of the least absolute differences of the corresponding coordinates of two points. (For our purposes, the underlying alphabet is the integers modulo m, and the "least absolute difference" between i and j in this alphabet is the smaller of $i - j \pmod m$ and $j - i \pmod m$.)

In general, a Lee sphere of radius r, in two dimensions, consists of $q = r^2 + (r + 1)^2 = 2r^2 + 2r + 1$ cells. The first few cases appear in Fig. 6. The obvious closed-packed codes to look for are close-packed codes of Lee radius r, with word length 2, over the q symbol alphabet, where $q = 2r^2 + 2r + 1$.

Such a code would correspond to a tiling of the $q \times q$ torus with polyominoes which are Lee spheres of radius r. The main result of this section is that such codes exist for all positive integers r.

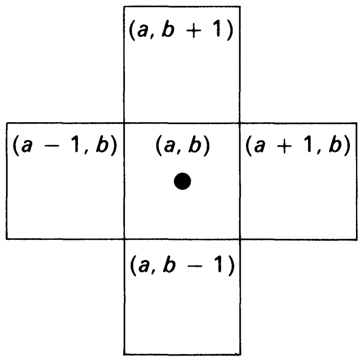

FIG. 5. *The X-pentomino as a Lee sphere of radius* 1

THEOREM 1. *For every positive integer r, there is a close-packed r-error-correcting dictionary in the Lee metric of codewords of length* 2, *over the q symbol alphabet*, $q = 2r^2 + 2r + 1$.

Note. Geometrically, this theorem asserts that q Lee spheres of radius r, in two dimensions, can be used to tile the $q \times q$ torus.

Proof. As codewords, we use the set $\{(a, (2r + 1)a)\}$ with $a = 0, 1, 2, \cdots, q - 1$, regarding all integers as modulo q. Since these codewords form a group under componentwise addition modulo q, the minimum distance between two codewords equals the minimum weight of any nonzero codeword, and the code is r-error-correcting if this minimum weight is at least $2r + 1$.

Consider $a \not\equiv 0 \pmod{q}$. If $\|a\| + \|(2r + 1)a\| \leqq 2r + 1$, then at least one of the two components contributes $\leqq r$. If $1 \leqq a \leqq r$, we have $(2r + 1)a < q$, so that the distance can be written

$$\begin{aligned}\|a\| + \|(2r + 1)a\| &= a + \min\,[(2r + 1)a, 2r^2 + 2r + 1 - (2r + 1)a] \\ &= \min\,[(2r + 2)a, 2r(r + 1 - a) + 1] \geqq 2r + 1,\end{aligned}$$

as required. It is not necessary to consider separately the case that the *second* component is $\leqq r$, since $(a, (2r + 1)a)$ can be rewritten $(-(2r + 1)b, b)$ where $b \equiv (2r + 1)a \bmod q$ and $a \equiv -(2r + 1)b \equiv -(2r + 1)^2 a \bmod q$. $(-(2r + 1)b, b)$ of course has the same weight as $(b, (2r + 1)b)$.

Since there are q nonoverlapping spheres with q points each, the $q \times q$ torus is covered and the code is close-packed.

For $r = 1$, the close-packing of the 5×5 torus with five X-pentominoes (spheres of Lee radius 1) is shown in Fig. 7. Note that the codewords (the centers

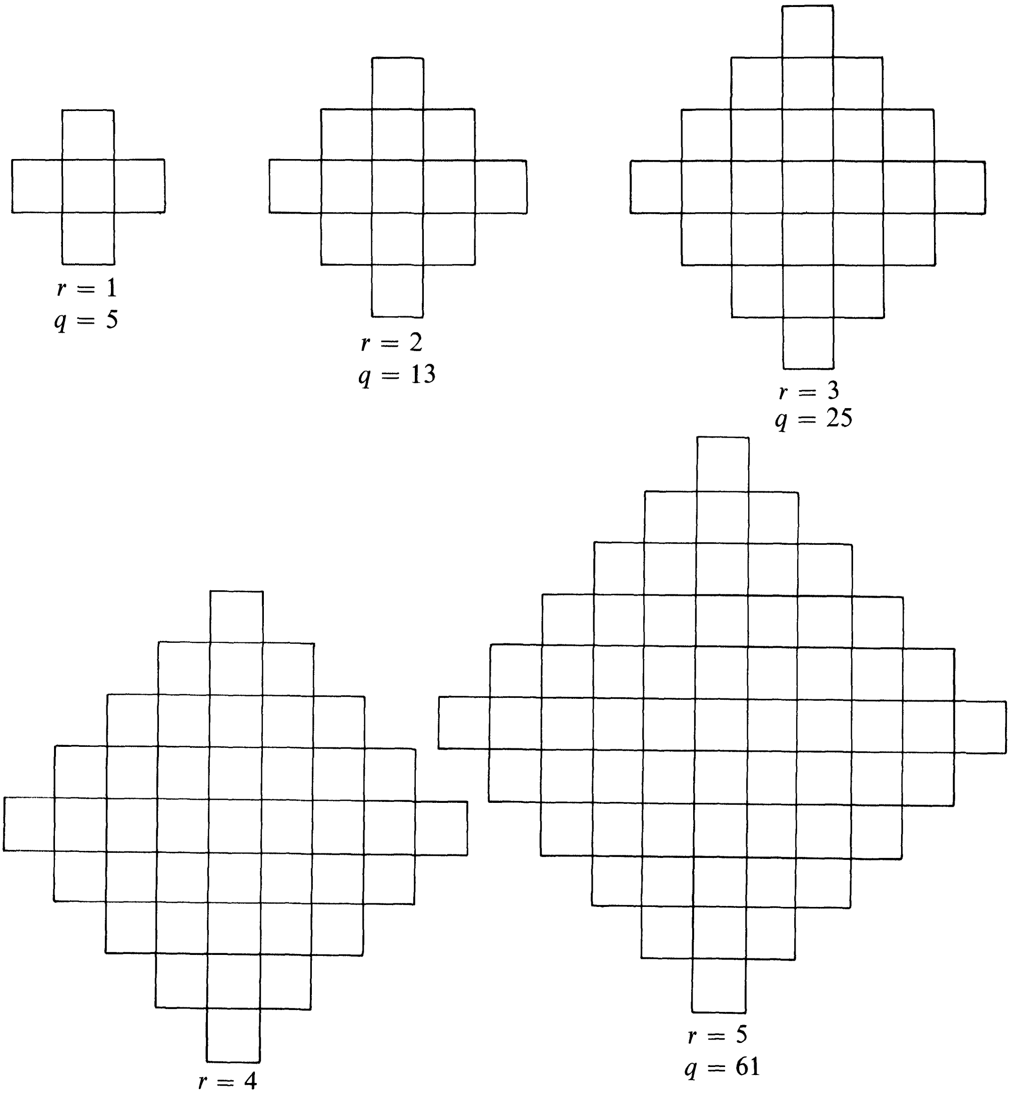

FIG. 6. *Two-dimensional Lee spheres of radius r for* $1 \leqq r \leqq 5$

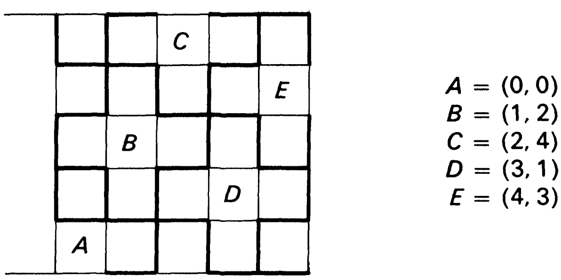

FIG. 7. *A close-packed code for the Lee metric, using X-pentominoes*

of the X's) are at the same positions as the codewords in Figs. 3 and 4. For $r = 2$, the close-packing of the 13×13 torus with thirteen triskaidekominoes (spheres of radius 2) is shown in Fig. 8.

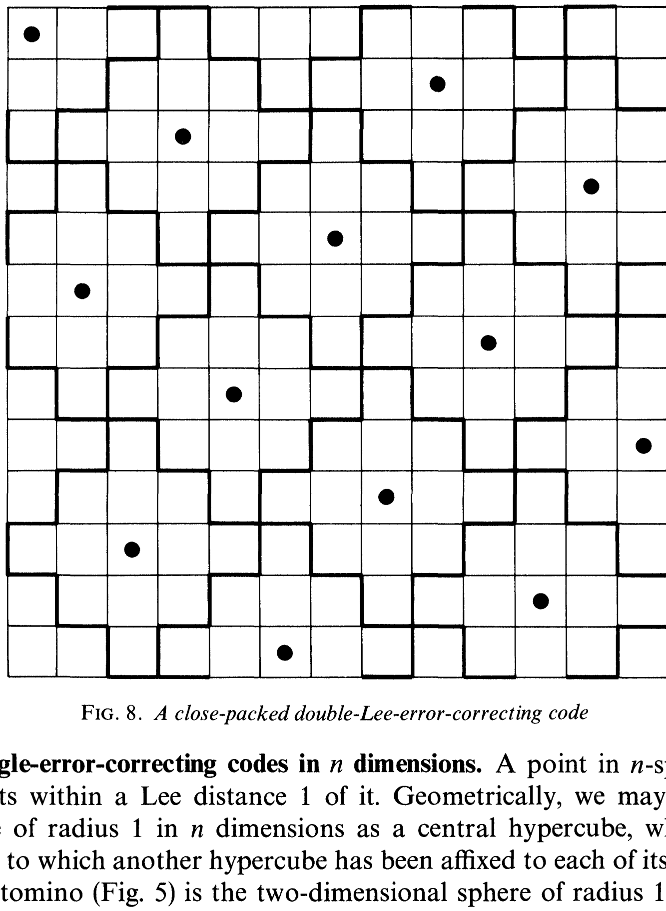

FIG. 8. *A close-packed double-Lee-error-correcting code*

3. Single-error-correcting codes in n dimensions. A point in n-space has $2n$ other points within a Lee distance 1 of it. Geometrically, we may visualize a Lee sphere of radius 1 in n dimensions as a central hypercube, which has $2n$ hyperfaces, to which another hypercube has been affixed to each of its hyperfaces. The X-pentomino (Fig. 5) is the two-dimensional sphere of radius 1. The three-dimensional sphere of radius 1 is the heptacube shown in Fig. 9. We can prove the following theorem directly.

THEOREM 2. 49 *of the heptacubes of Fig.* 9 *can be used to close-pack the* $7 \times 7 \times 7$ *hypertorus.*

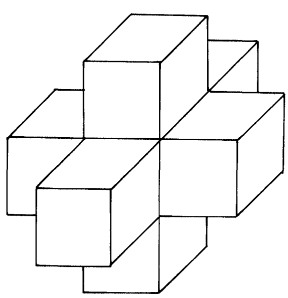

FIG. 9. *The* 3-*dimensional Lee sphere of radius* 1

Proof. Specifically, we look at a typical 7×7 cross section of the solution, shown in Fig. 10. The cross sections of our heptacube will be either X-pentominoes or single squares. In the cross section shown in Fig. 10, we see seven X-pentominoes, as well as seven squares labeled A and seven squares labeled B. The A's are bottoms of heptacubes whose centers are in the plane *above*, and the B's are tops of heptacubes protruding upward from the plane below. Since the seven A's are systematically translated (1 unit to the northwest) from the centers of the X-pentominoes, we are assured that in the next cross section above the one we are examining, the X-pentomino sections fit together properly. Similarly the seven B's are systematically translated (1 unit to the southeast) from the centers of the seven X-pentominoes and are therefore consistent as tops of heptacubes from the layer below. Finally, since 7 is a prime, it is easy to see that these translations must lead to a periodicity of 7 in the third dimension.

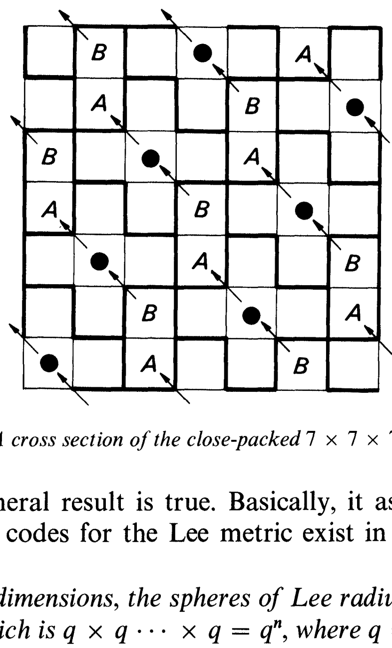

FIG. 10. *A cross section of the close-packed* $7 \times 7 \times 7$ *hypertorus*

A much more general result is true. Basically, it asserts that close-packed single error-correcting codes for the Lee metric exist in n dimensions, for all n, as follows.

THEOREM 3. *In n dimensions, the spheres of Lee radius* 1 *can be used to close-pack the hypertorus which is* $q \times q \cdots \times q = q^n$, *where* $q = 2n + 1$.

Proof. As centers of the spheres, we use the set S of all points $(a_1, a_2, \cdots, a_n)$ of the hypertorus which satisfy

$$\sum_{i=1}^{n} ia_i \equiv 0 \pmod{2n + 1}.$$

The number of solutions to this congruence is clearly q^{n-1}, since any choice of $a_2, a_3, \cdots, a_n$ may be made, and then there is a unique value of a_1, modulo q, to satisfy the congruence. Also, every point of the hypertorus is within a Lee distance of 1 from some point in this set. For if $B = (b_1, b_2, \cdots, b_n)$ is any point of the hypertorus, we compute

$$\sum_{i=1}^{n} ib_i \equiv k \pmod{2n + 1},$$

where $-n \leqq k \leqq +n$. If $k = 0$, then B is a member of the set S. If $k > 0$, we change b_k to $b_k - 1$ to go from B to a member of S at Lee distance 1 away. If $k < 0$, we change $b_{|k|}$ to $b_{|k|} + 1$ to go from B to a member of S at Lee distance 1 away.

Each point S has only $2n$ neighbors at a distance of 1 away. Thus the spheres around these points can account for at most $q^{n-1}(2n + 1) = q^n$ points if the spheres are all disjoint. However, since every point of the hypertorus is within distance 1 from some point of S, the spheres must be disjoint and fill up the space. Thus, the code is close-packed.

According to Theorem 1, for every positive integer r, the Lee sphere $S_{2,r}$ tiles 2-dimensional space. By Theorem 3, for every positive integer n, the Lee sphere $S_{n,1}$ tiles n-dimensional space. It is also trivially true that for every positive integer r, the Lee sphere $S_{1,r}$, which is merely a line segment of length $2r + 1$, tiles 1-dimensional space. These are the only cases for which tilings have been found, and we *conjecture* that no other cases exist. Partial results in support of this conjecture are contained in the last two sections of this paper.

4. Some special constructions. When $q = 2n + 1$ is a perfect power, it may be possible to construct a close-packed single-error-correcting code in the Lee metric, in n dimensions, with an alphabet size less than q. For example, when $n = 4$ and $q = 9$, rather than tiling the $9 \times 9 \times 9 \times 9$ hypertorus with the spheres of radius 1 composed of 9 tesseracts, as guaranteed by Theorem 3, we may attempt to tile the $3 \times 3 \times 3 \times 3$ hypertorus with such spheres.

A successful attempt at close-packing 9 of these spheres into the $3 \times 3 \times 3 \times 3$ hypertorus is shown in Fig. 11. The centers of the spheres are indicated by the boldface letters **A** through **I**. The other points of the sphere are indicated by the

		0			1			2		
		0	1	2	0	1	2	0	1	2
0	0	**A**	A	A	A	B	H	A	F	C
	1	A	B	D	B	**B**	B	I	B	C
	2	A	G	C	E	B	C	C	C	**C**
1	0	A	F	D	E	F	H	F	**F**	F
	1	D	D	**D**	E	B	D	I	F	D
	2	E	G	D	**E**	E	E	E	F	C
2	0	A	G	H	H	H	**H**	I	F	H
	1	I	G	D	I	B	H	**I**	I	I
	2	G	**G**	G	E	G	H	I	G	C

Codewords

A ~ (0, 0, 0, 0)
B ~ (0, 1, 1, 1)
C ~ (0, 2, 2, 2)
D ~ (1, 0, 1, 2)
E ~ (1, 1, 2, 0)
F ~ (1, 2, 0, 1)
G ~ (2, 0, 2, 1)
H ~ (2, 1, 0, 2)
I ~ (2, 2, 1, 0)

FIG. 11. *Close-packed four-dimensional code, single-error-correcting in both Hamming and Lee metrics*

same letter as the center, but in fainter type. The four coordinates of a point are its (a) superrow, (b) supercolumn, (c) subrow and (d) subcolumn.

Over the ternary alphabet, a single error in the Lee metric is the same as a single error in the Hamming metric. (If one component is in error, this is a single Hamming error, regardless of the *magnitude* of the error. For the cyclic ternary alphabet, the error in a component is necessarily by ± 1 modulo 3.) Thus, Fig. 11 is also a close-packed single-Hamming-error-correcting code! In fact, this code was obtained in [1] by the method of orthogonal Latin squares. Two orthogonal Latin squares of order n always lead to a single-Hamming-error-correcting code for word length 4 over the n symbol alphabet as follows:

We label the rows of the squares from 0 to $n - 1$, the columns from 0 to $n - 1$, and the entries are named 0 to $n - 1$. Then we form the set of all quadruples

$$(r, c, e_1, e_2),$$

where r is the row index, c is the column index, e_1 is the entry at the (r, c) position in the first square, and e_2 is the entry at the (r, c) position in the second square. It is easy to show that if (r, c, e_1, e_2) and (r', c', e_1', e_2') agree in any two of their

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

0	1	2
0	1	2
2	0	1
1	2	0

(r, c, e_1, e_2)
(0, 0, 0, 0)
(0, 1, 1, 1)
(0, 2, 2, 2)
(1, 0, 1, 2)
(1, 1, 2, 0)
(1, 2, 0, 1)
(2, 0, 2, 1)
(2, 1, 0, 2)
(2, 2, 1, 0)

FIG. 12. *From orthogonal Latin squares to a distance* 3 *code*

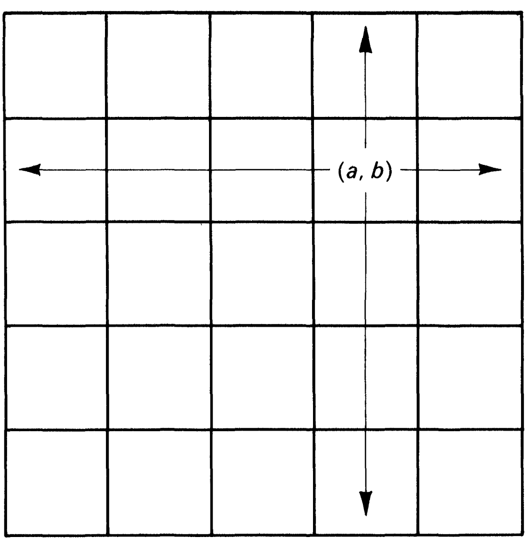

FIG. 13. *The rook domain of the square* (a, b)

components, then they must agree in all four. Hence, the set of all n^2 "points" (r, c, e_1, e_2) has a minimum Hamming distance of 3 between any two members and is therefore single-error-correcting.

The case $n = 3$ is illustrated in Fig. 12. The code obtained is the same as in Fig. 11.

In the Hamming metric, a "sphere of radius r" looks even less "round" than a sphere in the Lee metric. In [1], these Hamming-metric "spheres" are designated as *rook domains.* Specifically, in two dimensions, the single Hamming errors from the point (a, b) correspond to those squares to which a rook, located on the square (a, b), could go in a single move (see Fig. 13). In 2 dimensions, rook domains do not pack efficiently, but in higher dimensions they may. Besides the theory of rook-domain packing in [1], there is also a fundamental outstanding conjecture [2].

5. Sphere-packing constraints. If we denote by $V(n, r)$ the number of points contained in the n-dimensionsional sphere of Lee-radius r, it is rather easily established (see the proof of Theorem 4 below) that

$$V(n,r) = \sum_{k \geqq 0} 2^k \binom{n}{k}\binom{r}{k}.$$

It is a curious fact that $V(n, r)$ is symmetric in n and r. The effective upper limit of the summation is at $k = \min(n, r)$. By the usual sphere-packing argument, we obtain the following "sphere-packing bound."

THEOREM 4. *The number of codewords in an r-error-correcting code dictionary, for word length n and alphabet size q, where errors are measured in the Lee metric, cannot exceed*

$$\frac{q^n}{\sum_{k \geqq 0} 2^k \binom{n}{k}\binom{r}{k}}.$$

Proof. The codewords must be surrounded by disjoint spheres of radius r. There are q^n points in the sphere, and each codeword uses up $V(n, r)$ of them, so that there can be at most $q^n/V(n, r)$ codewords.

To establish the identity

$$V(n,r) = \sum_{k \geqq 0} 2^k \binom{n}{k}\binom{r}{k}$$

we regard the n components of a codeword as "boxes," and we have r "error balls" to distribute among these boxes. For any $k \leqq r$, we consider the problem of distributing up to r error balls into *exactly* k boxes. There are $\binom{n}{k}$ ways to choose k of the n boxes to contain all the balls; each of these k boxes must be designated as either containing a *positive* or *negative* deviation, for a factor of 2^k; and there are $\binom{r}{k}$ ways to distribute up to r balls into k boxes in such a way that no box is empty. Multiplying these three factors together, and then summing over k, leads directly to the formula for $V(n, r)$.

A *close-packed* code is one which attains the sphere-packing bound. Clearly, a necessary condition for a close-packed code to exist, for given n, r and q, is that $V(n, r)$ divide q^n. This necessary condition is met, in particular, when $q = V(n, r)$. However, as we shall see, $q = V(n, r)$ is both unnecessary and insufficient for a close-packed code to exist.

The underlying geometric problem is this: For what values of n and r does the n-dimensional sphere $S_{n,r}$ of radius r tile n-dimensional space? If the sphere is incapable of tiling the space, then no close-packed code can exist. If the sphere does tile the space, then any q such that the tiling is periodic with period q in each direction is an acceptable alphabet size.

The spheres $S_{1,r}$ and $S_{2,r}$ all succeed in filling their respective spaces, since $S_{1,r}$ is simply a line segment of length $2r + 1$, which fills up one-dimensional space periodically with a period of $q = 2r + 1$; and $S_{2,r}$ is the two-dimensional sphere of Theorem 1, which fills the plane periodically with a period of $q = 2r^2 + 2r + 1$.

Also, the spheres $S_{n,1}$ fill up n-dimensional space, according to Theorem 3, with a periodicity of $q = 2n + 1$. A smaller q (specifically, a factor of $2n + 1$ containing all the distinct prime factors of $2n + 1$) may sometimes be possible, as illustrated for $n = 4$, $q = 3$ in Fig. 11.

Not all spheres $S_{n,r}$ are capable of tiling n-dimensional space. The first counterexample is the following.

THEOREM 5. *The sphere $S_{3,2}$, illustrated in Fig.* 14 *and made up of* 25 *unit cubes, is unable to tile* 3-*space.*

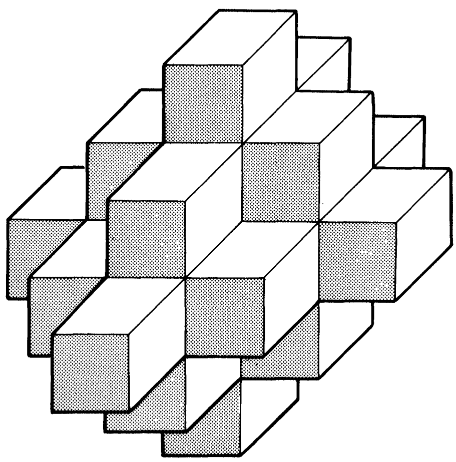

FIG. 14. *The Lee sphere $S_{3,2}$, composed of 25 unit cubes*

Proof. Let $S(a, b, c)$ be the Lee sphere of radius 2, dimension 3 and center (a, b, c). Assume that E^3 can be tiled and let

$$\{S(a_i, b_i, c_i) | i = 0, 1, 2, \cdots\}$$

be a tiling. We may also assume $(a_0, b_0, c_0) = (0, 0, 0)$.

Let $S(a_1, b_1, c_1)$ be the sphere containing $(2, 1, 0)$, so that

$$|a_1 - 2| + |b_1 - 1| + |c_1| \leqq 2.$$

Since $S(a_1, b_1, c_1)$ and $S(0, 0, 0)$ are disjoint,

$$5 \leqq |a_1| + |b_1| + |c_1|.$$

However, by the triangle inequality,

$$|a_1| + |b_1| + |c_1| \leqq |a_1 - 2| + 2 + |b_1 - 1| + 1 + |c_1| \leqq 2 + 3 = 5,$$

with equality holding if and only if $a_1 \geqq 2, b_1 \geqq 1$. It follows that $a_1 \geqq 2$ and $b_1 \geqq 1$ and $a_1 + b_1 + |c_1| = 5$.

If $a_1 \geqq 3$, then

$$|a_1 - 3| + |b_1| + |c_1| = a_1 - 3 + b_1 + |c_1| = 2$$

and $(3, 0, 0) \in S(a_1, b_1, c_1)$.

The point $(2, -1, 0)$ is outside S_0 and S_1 and therefore in a third sphere S_2. An argument similar to the one above shows that $a_2 \geqq 2$, and if $a_2 \geqq 3$ then $(3, 0, 0) \in S_2$. Since S_1 and S_2 are disjoint, either $a_1 = 2$ or $a_2 = 2$. Using a symmetry of E^3, we may assume $a_1 = 2$ and $c_1 \geqq 0$, and consider the three cases:

(a) $(a_1, b_1, c_1) = (2, 3, 0)$,

(b) $(a_1, b_1, c_1) = (2, 2, 1)$,

(c) $(a_1, b_1, c_1) = (2, 1, 2)$.

Again a symmetry of E^3 can be used to reduce case (c) to case (b).

Case (a). $S_0 = S(0, 0, 0)$ and $S_1 = S(2, 3, 0)$ are members of a tiling. Since the point $(1, 1, 1)$ is not in S_0 or S_1, it must be in another S, say $S(a_2, b_2, c_2)$. We have

$$\begin{aligned} &|a_2| + |b_2| + |c_2| \geqq 5, \\ &|a_2 - 2| + |b_2 - 3| + |c_2| \geqq 5, \\ &|a_2 - 1| + |b_2 - 1| + |c_2 - 1| \leqq 2. \end{aligned} \tag{1}$$

The only solution to these inequalities is

$$(a_2, b_2, c_2) = (1, 1, 3).$$

Next, consider the point $(1, 2, 1)$ and the isometric linear transformation of E^3 $\varphi(x, y, z) = (2 - x, 3 - y, z)$. This maps $(1, 2, 1)$ into $(1, 1, 1)$ and interchanges S_0 and S_1. Therefore the center of the sphere S_3 containing $(1, 2, 1)$ is $\varphi^{-1}(1, 1, 3) = (1, 2, 3)$. S_2 and S_3 are neither disjoint nor identical, contrary to the initial hypothesis.

Case (b). $S_0 = S(0, 0, 0)$ and $S_1 = S(2, 2, 1)$ are members of a tiling. Consider the point $(1, 1, -1)$. Using an argument similar to Case (a) we have

$$\begin{aligned} &|a_2| + |b_2| + |c_2| \geqq 5, \\ &|a_2 - 2| + |b_2 - 2| + |c_2 - 1| \geqq 5, \\ &|a_2 - 1| + |b_2 - 1| + |c_2 + 1| \leqq 2. \end{aligned} \tag{2}$$

The only solution to these inequalities is

$$(a_2, b_2, c_2) = (1, 1, -3).$$

The point $(1, 2, -1)$ is not in S_0, S_1 or S_2 and therefore in some S_3. We obtain the inequalities

$$\begin{aligned} &|a_3| + |b_3| + |c_3| \geqq 5, \\ &|a_3 - 2| + |b_3 - 2| + |c_3 - 1| \geqq 5, \\ &|a_3 - 1| + |b_3 - 1| + |c_3 + 3| \geqq 5, \\ &|a_3 - 1| + |b_3 - 2| + |c_3 + 1| \leqq 2. \end{aligned} \tag{3}$$

The second and fourth inequalities imply

$$a_3 \leqq 1, \qquad c_3 \leqq -1,$$

while the third and fourth imply

$$b_3 \geqq 2, \quad c_3 \geqq -1 \quad \text{and} \quad |a_3| + b_3 - 2 + |c_3 + 3| = 5.$$

With this information, the first and fourth then imply the unique solution

$$(a_3, b_3, c_3) = (1, 4, -1).$$

Next, let φ be the mapping of E^3 where $\varphi(x, y, z) = (z + 3, y - 1, x - 1)$. The tiling produced by applying φ to the hypothesized tiling has, as members, $\varphi S_2 = S(0, 0, 0)$ and $\varphi S_3 = S(2, 3, 0)$. But this is Case (a) which has already been shown to yield a contradiction.

6. The equivalent tessellation with cross-polytopes. A general proof of the inability of $S_{n,r}$ to tile n-dimensional space, for large classes of n and r, can be based on the approximation of $S_{n,r}$ by the n-dimensional cross-polytope.

DEFINITION. For every Lee sphere $S_{n,r}$, we define the *conscribed cross-polytope* to be the smallest convex figure containing the 2^n center points of its $(n - 1)$-dimensional extremal hyperfaces.

In Fig. 15, the conscribed cross-polytopes are illustrated in 2 and 3 dimensions. In 2 dimensions the figure is a square, and in 3 dimensions, a regular octahedron. In

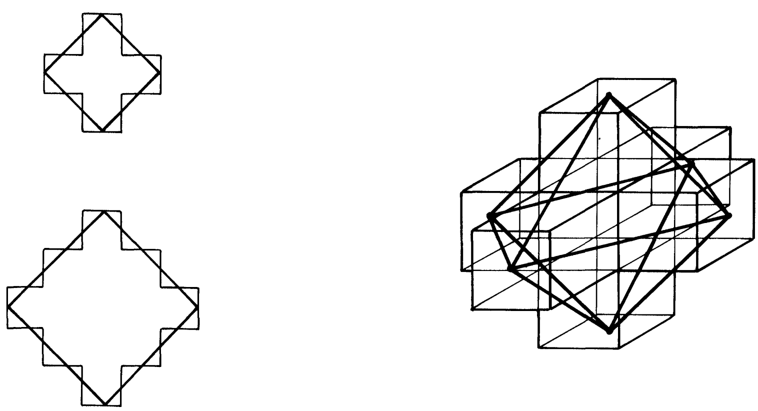

FIG. 15. *Examples of the conscribed cross-polytopes in* 2 *and* 3 *dimensions*

n dimensions, it is the regular cross-polytope, of n-dimensional hypervolume

$$V_{CP}(n,r) = \frac{d^n}{n!} = \frac{(2r+1)^n}{n!},$$

where $d = 2r + 1$ is the Euclidean diameter of $S_{n,r}$.

The significant fact about these figures is that any packing of n-dimensional space with the spheres $S_{n,r}$ induces a (less efficient) packing with the conscribed cross-polytopes. In general, the relative efficiency factor is

$$\frac{V_{CP}(n,r)}{V(n,r)},$$

which is less than unity whenever $n > 1$.

In Fig. 16 we see a tiling of the plane with the X-pentomino ($S_{2,1}$) and the induced tessellation with conscribed squares. The efficiency of this square tiling is

$$\frac{V_{CP}(2,1)}{V(2,1)} = \frac{9/2}{5} = 90\%,$$

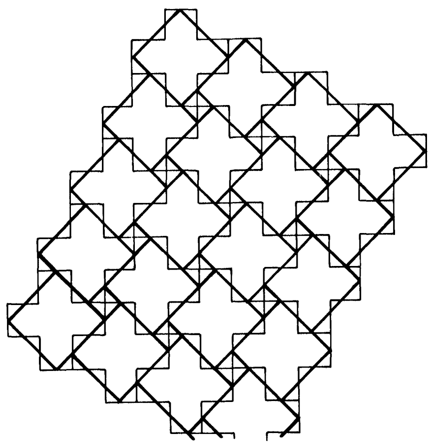

FIG. 16. *The conscribed square tiling induced by the* $S_{2,1}$ *tiling*

and we observe that for each square of area 9/2 conscribed in a pentomino, there is a left-over square of area 1/2.

We use this type of argument to prove the following two theorems.

THEOREM 6. *The sphere* $S_{3,r}$ *cannot tile* 3-*space for any* $r > r_0$.

Proof. If $S_{3,r}$ tiles 3-space, it induces a packing of 3-space with (conscribed) regular octahedra, with a packing efficiency of

$$E_3(r) = \frac{V_{CP}(3,r)}{V(3,r)} = \frac{(2r+1)^3/6}{(8r^3+12r^2+16r+6)/6} = \frac{(2r+1)^3}{(2r+1)^3+5(2r+1)}$$
$$= \frac{1}{1+5/(2r+1)^2}.$$

Now it is known [3] that equal regular octahedra are not capable of completely filling 3-dimensional space. It can be shown (see the Appendix) that if a figure does not fill space with a packing efficiency of unity, then there is an upper bound α to the packing density, with $\alpha < 1$. (For the octahedron, there is an obvious construction with a packing efficiency of 2/3; and we have now located a reference [5] for the best possible α.) As soon as E_3 exceeds α, the Lee sphere packing induces an octahedral packing which exceeds the limit on octahedral packing density. Since $E_3(r) \to 1$ as $r \to \infty$, $E_3(r) > \alpha$ for $r > r_0$.

This is readily generalized in the following theorem.

THEOREM 7. *For $n > 4$ and $r > \rho_n$, the sphere $S_{n,r}$ cannot tile n-space.*

Proof. In n-dimensional Euclidean space, for $n > 4$, it is known [3] that the regular cross-polytope does not tile the space. Again, there is a maximum packing density α_n, which would be exceeded by the conscribed spheres, for $r > \rho_n$, if the Lee sphere packing existed. This depends only on the fact that

$$E_n(r) = \frac{V_{CP}(n, r)}{V(n, r)} \to 1 \quad \text{as } r \to \infty.$$

Here, references to limiting packing densities for the cross-polytopes have not been found. For $n = 3$, however, $\alpha = 18/19$, by [5].

7. Summary. The Lee spheres $S_{n,r}$ are found to tile n-dimensional Euclidean space, in closed-packed fashion, when

(a) $n = 1$ for all r,
(b) $n = 2$ for all r,
(c) $r = 1$ for all n.

It is conjectured that these are the only cases for which a close-packing exists. The close-packing has been proved *not* to exist when

(a) $n = 3, r = 2$,
(b) $n > 4, r > \rho_n$, where ρ_n depends on the limit to packing efficiency of the cross-polytope in n -dimensional Euclidean space.

Appendix. The proof of Theorem 6 used the fact that there exists a tiling whose packing efficiency is the supremum of all packing efficiencies. A general form of this fact is now proved.

Let $A_1, \cdots, A_k$ be bounded measurable sets in R^n, the n-dimensional vector space over the reals. A sequence of ordered pairs,

$$T = \{(p_i, k_i) : i = 1, \cdots\}$$

is a tiling provided the p_i are points of R^n, the k_i are positive integers less than or equal to k, and the sets

$$A_{k_i} + p_i$$

are pairwise disjoint to within measure zero.

Let C_r by the hypercube of side $2r$ and center at the origin. Let A_T be the union of all tiles in T, that is,

$$A_T = \bigcup \{A_k + p : (p, k) \in T\},$$

and let

$$\alpha_T = \limsup_{r\to\infty} \frac{\mu(A_T \cap C_r)}{\mu(C_r)},$$

where μ is Lebesque measure in R^n.

THEOREM. *There exists a tiling* T_0 *with*

$$\alpha_{T_0} = \sup_T \alpha_T.$$

Proof. Let $A_{T,r}$ be the union of all tiles in T which are completely in C_r, that is,

$$A_{T,r} = \bigcup\{A_k + p:(p, k) \in T, \mu[\bar{C}_r \cap (A_k + p)] = 0\}.$$

Because of the boundedness of the A_i,

$$\alpha_T = \limsup_{r\to\infty} \frac{\mu(A_{T,r})}{\mu(C_r)}.$$

The construction of T_0 is as follows. For each positive integer i there exists a tiling T_i such that

$$\alpha_{T_i} > (\sup_T \alpha_T) - 2^{-i}.$$

From the definition of lim sup, there exists a sequence of numbers r_i such that

$$\frac{\mu(A_{T_i}, r_i)}{\mu(C_{r_i})} > \alpha_{T_i} - 2^{-i}$$

and

$$r_i > 2^i(r_{i-1} + B),$$

where B is a bound on the maximum distance of the points of the A_i from the origin. Finally define T_0 as the sequence which, for each i, contains those tiles $A + x$ of T_i which are entirely in $C_{r_i} - C_{r_{i-1}}$, that is, which satisfy

$$\mu[(A + x) \cap \bar{C}_{r_i}] = 0,$$

$$\mu[(A + x) \cap C_{r_{i-1}}) = 0.$$

It can be shown that T_0 is a tiling and that

$$\mu(A_{T_0,r_i}) \geqq \mu(A_{T_i,r_i}) - \mu(C_{r_{i-1}+B}).$$

It follows that

$$\frac{\mu(A_{T_0,r_i})}{\mu(C_{r_i})} \geqq (\sup_T \alpha_T) - \frac{1}{2^i} - \frac{1}{2^i} - \frac{1}{2^{ni}}$$

and

$$\alpha_{T_0} = \limsup_{r\to\infty} \frac{\mu(A_{T_0,r})}{\mu(r)} \geqq \sup_T \alpha_T.$$

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