PERFECT CODES FROM PGL(2,5) IN STAR GRAPHS

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Abstract. The Star graph S_n is the Cayley graph of the symmetric group Sym_n with the generating set $\{(1\ i) : 2 \leq i \leq n\}$. Arumugam and Kala proved that $\{\pi \in \mathrm{Sym}_n : \pi(1) = 1\}$ is a perfect code in S_n for any $n, n \geq 3$. In this note we show that for any $n, n \geq 6$ the Star graph S_n contains a perfect code which is a union of cosets of the embedding of $\mathrm{PGL}(2,5)$ into Sym_6.

Keywords: perfect code, efficient dominating set, Cayley graph, Star graph, projective linear group, symmetric group.

1. Introduction

Let G be a group with an inverse-closed generating set H that does not contain the identity. *The Cayley graph* $\Gamma(G, H)$ is the graph whose vertices are the elements of G and the edge set is $\{(hg, g) : g \in G, h \in H\}$. The *symmetric group* of degree n is denoted by Sym_n. The stabilizer of an element $i \in \{1, \ldots, n\}$ by Sym_n is denoted by $Stab_i(\mathrm{Sym}_n)$. *The Star graph* S_n is $\Gamma(\mathrm{Sym}_n, \{(1\ i) : 2 \leq i \leq n\})$.

A *code* in a graph G is a subset of its vertices. The *size* of C is $|C|$. The *minimum distance* of a code is $d = min_{x,y \in C, x \neq y} d(x, y)$, where $d(x, y)$ is the length of a shortest path connecting x and y. A code C is *perfect* (also known as efficient dominating set) in a k-regular graph Γ with vertex set V if it has minimum distance 3 and the size of C attains *the Hamming upper bound*, i.e. $|C| = |V|/(k + 1)$. We say that two codes in a graph Γ are *isomorphic* if there is an automorphism of the graph Γ that maps one code into another.

Let T_0, T_1 be distinct subsets of vertices of a graph Γ. The ordered pair (T_0, T_1) is called a *perfect bitrade*, if for any vertex x, the set consisting of x and its neighbors in Γ meets T_0 and T_1 in the same number of vertices that is zero or one. The size of $|T_0|$ is called the *volume* of the bitrade. In particular, if C and C' are perfect codes in Γ, then $(C \setminus C', C' \setminus C)$ is a perfect bitrade. In this case the bitrade $(C \setminus C', C' \setminus C)$ is called *embeddable* into a perfect code. In general, bitrades (non necessarily perfect) are often associated with classical combinatorial objects such as perfect codes, Steiner triple and quadruple systems and latin squares (e.g. see a survey [10]). Bitrades are used in constuctions of the parent combinatorial objects or for obtaining upper bounds on their number.

The first well-known error-correcting code was the binary Hamming code. This code is a perfect code in the Hamming graph, which is a Cayley graph of the group

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Z_2^n. Later in [14] Vasiliev showed that there are perfect codes that are nonisomorphic to the Hamming codes. A somewhat similar fact holds for the Star graph as in Section 3 we show that there are perfect codes nonisomorphic to the first series of perfect codes in the Star graph from [3].

Generally speaking, the permutation codes are subsets of Sym_n with respect to a certain metric. These codes are of practical interest for their various applications in areas such as flash memory storage [13] and interconnection networks [1]. The permutation codes with the Kendall τ-metric (i.e. codes in the *bubble-sort graph* $\Gamma(\mathrm{Sym}_n, \{(i\ i+1) : 1 \leq i \leq n-1\})$) were considered by Etzion and Buzaglo in [11]. They showed that no perfect codes in these graphs exist when n is prime or $4 \leq n \leq 10$. In [12] the nonexistence of the perfect codes in the Cayley graphs $\Gamma(\mathrm{Sym}_n, H)$ was established, where H are transpositions that form a tree of diameter 3.

The spectral graph theory is important from the point of view of coding theory. In particular, according to the famous Lloyd's theorem the existence of a perfect code in a regular graph necessarily implies that -1 is an eigenvalue of the graph. The integrity of the spectra of several classes of Cayley graphs of the symmetric and the alternating groups was proven in [7]. The eigenvalues of S_n are all integers $i, -(n-1) \leq i \leq (n-1)$ that follows from the spectra of the Jucys-Murphy elements [8]. The multiplicities of the eigenvalues of S_n were studied in [2] and the second largest eigenvalue $n-2$ was shown to have multiplicity $(n-1)(n-2)$. In [5] an explicit basis for the eigenspace with eigenvalue $n-2$ was found and a reconstruction property for eigenvectors by its partial values was proven. Later in [6] it is shown that the basis consists of eigenvectors with minimum support.

For $l, r \in \mathrm{Sym}_n$ define the following mapping on the vertices of S_n: $\lambda_{l,r}(g) = lgr$, g in Sym_n.

Theorem 1. [9] *The automorphism group of S_n is $\{\lambda_{l,r} : l \in Stab_1(\mathrm{Sym}_n), r \in \mathrm{Sym}_n\}$.*

In [3] Arumugam and Kala showed that $Stab_1(\mathrm{Sym}_n)$ is a perfect code in S_n, for any $n \geq 3$. Consider the isomorphism class of $Stab_1(\mathrm{Sym}_n)$ in S_n. By Theorem 1 the only left multiplication automorphisms are those by the elements from $Stab_1(\mathrm{Sym}_n)$. Therefore we have the following result.

Corollary 1. *The isomorphism class of $Stab_1(\mathrm{Sym}_n)$in S_n is the set of its right cosets in Sym_n.*

In Section 2 we prove that the projective linear group $\mathrm{PGL}(2,5)$ is a perfect code, which is isomorphic to $\{\pi \in \mathrm{Sym}_6 : \pi(1) = 1\}$ as a group via an outer automorphism of Sym_6, but is nonisomorphic to it with respect to the automorphism group of the Star graph. We continue the study in Section 3 where we construct a new series of perfect codes in Star graphs S_n, $n \geq 7$ using cosets of $\mathrm{PGL}(2,5)$. Also we obtain the classification of the isomorphism classes of perfect codes and perfect bitrades in Star graphs S_n, $n \leq 6$ by linear programming.

2. Perfect codes from $\mathrm{PGL}(2,5)$ in S_6

The action of a group G on a set M is *regular* if it is transitive and $|G| = |M|$, i.e. for any x, $y \in M$ there is exactly one element of G sending x to y.

Let $\mathrm{PGL}(n,q)$ be the projective linear group induced by the action of $\mathrm{GL}(n,q)$ on the 1-dimensional subspaces (projective points) of a n-dimensional space over the field of order q. It is well known that $\mathrm{PGL}(n, q)$ acts transitively on the ordered pairs of distinct projective points for $n \geq 3$ and regularly on the ordered triples of pairwise distinct projective points when $n = 2$, see e.g. [4][Exercises 2.8.2 and 2.8.7].

Proposition 1. *The group* $\mathrm{PGL}(2, q)$ *acts regularly on the ordered triples of distinct projective points.*

In throughout what follows we enumerate the projective points by the elements of $\{1, \dots, 6\}$, so $\mathrm{PGL}(2, 5)$ is embedded in Sym_n, $n \geq 6$. An element of Sym_n is a *cycle* of length m, if it permutes $i_1, \dots, i_m \in \{1, \dots, n\}$ in the cyclic order and fixes every element of $\{1, \dots, n\} \setminus \{i_1, \dots, i_m\}$.

Corollary 2. *The group* $\mathrm{PGL}(2, 5)$ *does not contain cycles of length* 2 *or* 3*.*

Proof. By Proposition 1 the group $\mathrm{PGL}(2, 5)$ is regular on the triples of the elements of $\{1, \dots, 6\}$. In particular, any permutation of $\mathrm{PGL}(2, 5)$ that has at least three fixed projective points is the identity. We conclude that there are no cycles of length 2 or 3 in $\mathrm{PGL}(2, 5)$ since they have three fixed points. □

Lemma 1. *Let* π *be a permutation from* Sym_n*,* $n \geq 6$*. Then* $\pi\mathrm{PGL}(2, 5)$ *is a code in* S_n *with the minimum distance 3.*

Proof. Suppose that $\pi\pi'$ and $\pi\pi''$ are adjacent in S_n, π', $\pi'' \in \mathrm{PGL}(2, 5)$. Then by the definition of the Star graph S_n there is x, $2 \leq x \leq n$ such that $(1\ x)\pi\pi' = \pi\pi''$, so $\pi^{-1}(1\ x)\pi = \pi''(\pi')^{-1}$ is in $\mathrm{PGL}(2, 5)$. This contradicts Corollary 2 because $\pi^{-1}(1x)\pi$ is a transposition. If $\pi\pi'$ and $\pi\pi''$ are at distance 2 in S_n, then there are x and y, $2 \leq x, y \leq n$, $x \neq y$ such that $\pi^{-1}(1\ x)(1\ y)\pi$ is in $\mathrm{PGL}(2, 5)$. So, $\pi^{-1}(1\ x)(1\ y)\pi$ is a cycle of length 3, which contradicts Corollary 2.

□

Theorem 2. *The group* $\mathrm{PGL}(2, 5)$ *is a perfect code in* S_6 *and the partitions of* Sym_6 *into the left and into the right cosets by* $\mathrm{PGL}(2, 5)$ *are partitions of the Star graph* S_6 *into perfect codes.*

Proof. The order of $\mathrm{PGL}(2, 5)$ is $5!$, which is the size of a perfect code in S_6 by the Hamming bound. Lemma 1 implies that $\mathrm{PGL}(2, 5)$ as well as any left coset of $\mathrm{PGL}(2, 5)$ is a perfect code. Since the right multiplication by any element of S_n is an automorphism of S_n by Theorem 1, every right coset of $\mathrm{PGL}(2, 5)$ is also a perfect code. The partitions into the left and right cosets are different because $\mathrm{PGL}(2, 5)$ is not a normal subgroup in Sym_6.

□

3. Recursive construction for perfect codes in the Star graphs from $\mathrm{PGL}(2, 5)$

Let C be a code in S_n. For a permutation σ from $Sym(n)$ denote by $\sigma C = \{\sigma\pi : \pi \in C\}$. If σ fixes 1 by Theorem 1 the left multiplication by σ is an automorphism of S_n and therefore the set of distances between any two permutations of C coincides with that of σC. In this section we show that a code in the Star graph S_{n-1} with minimum distance three could be embedded into a code in the Star graph S_n with minimum distance three by taking $(n-1)$ left multiplications of C by transpositions. In particular, we obtain a new infinite series of perfect codes in the Star graphs S_n from $\mathrm{PGL}(2, 5)$ for any $n, n \geq 6$.

Theorem 3. *Let C be a code with minimum distance 3 in S_{n-1}. Then the code*

$$C^n = C \cup \bigcup_{2\le i\le n-1} (i\ n)C$$

is a code of size $|C|(n-1)$ with minimum distance 3.

Proof. We introduce an auxilary notation and prove a technical result. Let Γ_i denote the subgraph of S_n induced by the set of vertices $(i\ n)\mathrm{Sym}_{n-1}$, $i \in 1,\dots,n-1$, Γ_n denote the subgraph of S_n induced by the vertices from Sym_{n-1}. Note that in [5] (see also [6][Section 6]) a similar partition was considered for constructing a basis for eigenspace of S_n corresponding to eigenvalue $n-2$.

Lemma 2. *1. For any $i, 2 \le i \le n$, Γ_i is an isometric subgraph of S_n that is isomorphic to S_{n-1}. The set of vertices of Γ_1 is a perfect code in S_n.*

2. Let π be a permutation from Sym_{n-1}. Then for any i, $2 \le i \le n-1$ the vertex $(i\ n)\pi$ of Γ_i has exactly one neighbor in S_n outside of Γ_i and it is the vertex (1 n)(1 i) of Γ_1. The only neighbor of π in S_n outside Γ_n is $(1\ n)\pi$.

Proof. 1. Obviously, the vertices of Sym_{n-1} induce an isometric subgraph of S_n which is isomorphic to S_{n-1}. By Theorem 1 the left multiplication by $(i\ n)$ is an automorphism of S_n for any $i \in \{2,\dots,n\}$. We conclude that Γ_i are isomorphic copies of S_{n-1} for any $i \in \{2,\dots,n\}$. By Corollary 1 we have that $(Stab_1(\mathrm{Sym}_n))(1\ n) = (1\ n)\mathrm{Sym}_{n-1}$ is a perfect code in S_n. Since this set is exactly the vertices of Γ_1, we obtain the required.

2. Since Γ_i is isomorphic to S_{n-1}, it is $(n-2)$-regular for $i \in \{2,\dots,n-1\}$. The remaining neighbor of $(i\ n)\pi$ outside Γ_i is the vertex $(1\ i)(i\ n)\pi = (1\ n)(1\ i)\pi$ of Γ_1. □

Obviously, the size of C^n is $(n-1)|C|$. We now show that the minimum distance of C^n is three. We see that each of the graphs Γ_i contains the copy $(i\ n)C$ of the code C, for any $i \in \{2,\dots,n-1\}$ and Γ_n contains C. The distances between vertices from $(i\ n)C$ are the same as those of C in S_{n-1}. Therefore, it remains to show that the distances between the vertices of $(i\ n)C$ and $(k\ n)C$ and the distances between the vertices of $(i\ n)C$ and C are at least 3, for any distinct i, k such that $2 \le i, k \le n-1$. By the second statement of Lemma 2, these distances are at least 2.

Let $(i\ n)\pi$ and $(k\ n)\pi'$ be at distance 2, $\pi, \pi' \in C$. Then by the second statement of Lemma 2 they both have a common neighbor in Γ_1, which is $(1\ n)(1\ i)\pi = (1\ n)(1\ k)\pi'$. This implies that $(1\ i)(1\ k)\pi' = \pi$ for $1 \le i, k \le n-1$, or equivalently π and π' are at distance 2 in S_{n-1}. This contradicts the minimum distance of C.

Let $(i\ n)\pi$ and π' be at distance 2, $\pi, \pi' \in C$. By the second statement of Lemma 2 the only neighbor of $(i\ \pi)$ outside of Γ_i is $(1\ n)(1\ i)\pi$ and the only neighbor of π' outside Γ_n is $(1\ n)\pi'$. So we see that $(1\ n)(1\ i)\pi = (1\ n)\pi'$, which contradicts the minimum distance of C.

□

Corollary 3. *For any $n \ge 6$ there is a perfect code in S_n which is not isomorphic to $Stab_1(Sym_n)$.*

Proof. Consider the code D which is obtained by iteratively applying construction from Theorem 3 $(n-6)$ times to the code $\mathrm{PGL}(2,5)$. By the construction, the code $\mathrm{PGL}(2,5)$ is a subcode of D. Proposition 1 implies that there are permutations

π, π' in $\mathrm{PGL}(2,5)$ such that $\pi(1) \neq \pi'(1)$. By Corollary 1 the isomorphism class of $Stab_1(Sym_n)$ in S_n consists of its right cosets. Since we have that $\pi(1) = \pi'(1)$ for any π and π' from a right coset of $Stab_1(Sym_n)$, we conclude that D is not isomorphic to $Stab_1(Sym_n)$. □

We proceed with the following computational results for small Star graphs.

Proposition 2. *1. The isomorphism class of $Stab_1(Sym_n)$ is the only isomorphism class of the perfect codes in S_n for n=3,4,5.*
2. The isomorphism classes of $Stab_1(Sym_6)$ and $\mathrm{PGL}(2,5)$ are the only isomorphism classes of the perfect codes in S_6.

Proof. For $n = 3$ and 4 the uniqueness of perfect code in S_n could be shown by hand. In case when $n = 5$ and 6 the result was obtained by binary linear programming. Because S_n is a transitive graph, without restriction of generality, we can consider the perfect codes containing the identity permutation. In case $n = 5$ there is one solution to the binary linear programming problem, which is $Stab_1(Sym_n)$.

Let n be six. We consider any transposition that preserves 1, say (2 3). By the definition of the Star graph, (2 3) is at distance three from the identity permutation. Now we split the set of all codes as follows: the codes that contain the permutation (2 3) and those that do not. We then solve two linear programming problems separately for these cases. There are 6 solutions (perfect codes) that does not contain (2 3). These are $\mathrm{PGL}(2,5)$ and its five conjugations. When (2 3) is in the code, the returns with the only solution which is $Stab_1(Sym_n)$.

□

Proposition 3. *All perfect bitrades in S_n are embeddable for $3 \leq n \leq 6$. For $n \in \{3,4,5\}$ their volumes are equal to $(n-1)!$. For $n = 6$ the volumes of bitrades are 120, 100 and 96.*

Proof. The statement is obvious for $n = 3$. Using linear programming approach by PC we found that for $n = 4,5,6$ all bitrades are embeddable and have the corresponding volumes. When n is 6, a perfect bitrade $(C \setminus C', C' \setminus C)$ has volume 120 if C and C' are disjoint perfect codes, e.g. $Stab_1(\mathrm{Sym}_6)$ and $Stab_1(\mathrm{Sym}_6)(1\ 6)$. By Proposition 1 the group $\mathrm{PGL}(2,5)$ acts transitively on the set $\{1,\dots,6\}$, so there are exactly 20 permutations from $\mathrm{PGL}(2,5)$ that fix 1. So we see that a perfect bitrade $(C \setminus C', C' \setminus C)$ is of volume 100 if C is $Stab_1(\mathrm{Sym}_6)$ and C' is $\mathrm{PGL}(2,5)$. Finally, $(C \setminus C', C' \setminus C)$ is a perfect bitrade of volume 96 if C is $\mathrm{PGL}(2,5)$ and C' is one of its nontrivial conjugations. Indeed, $\mathrm{PGL}(2,5)$ is isomorphic to Sym_5 via an outer automorphism of Sym_6. Therefore the intersection of $\mathrm{PGL}(2,5)$ and its conjugation is a subgroup which is isomorphic to the intersection Sym_5 and some of its conjugation $Stab_i(\mathrm{Sym}_6)$, $i \in \{1,\dots,5\}$. Since the latter intersection is of order 4!=24, the proposition is true.

□

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