Optimized Permutation Modulation

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Abstract —**Permutation modulation is a class of group codes for the Gaussian channel. A group code is a set of vectors obtained by a group of orthogonal matrices operating on an initial vector. An important problem is to find the initial vector yielding the largest minimum Euclidean distance between two vectors. This problem is solved for permutation modulation. We have also analyzed the performance of such optimized permutation modulation. Compared to binary antipodal signals and PSK it requires typically 2.5–4 dB less energy for the same minimum distance.**

I. Introduction

SIGNALS on a channel disturbed by additive white Gaussian noise (AWGC) are conventionally described in a Euclidean vector space [1]. A code for the AWGC is a set of M signals, equivalently described as a set of M vectors in a Euclidean vector space. The code is invariant under multiplication with a group G of orthogonal matrices. Thus multiplication with G transforms any one of the code vectors into a subset of the code vectors. The code is defined as a *group code for the AWGC* [2] if there is a set of M matrices in G such that all of the vectors can be obtained from a given initial vector by multiplication with the set of M matrices. This introduces a symmetry in the code. The probability of error, for example, when a maximum likelihood detector is used, is independent of the sent signal. Most signal sets used in practice can be described as group codes for the AWGC, for example all linear binary codes, the symbols of which are transmitted as antipodal ($+$ or -1) signals.

A special case of group codes for the AWGC is permutation modulation [3]. Slepian defined two types. In Variant 1 the code is the set of vectors obtained by permuting the components of a given initial vector in all possible ways. In Variant 2 the code is obtained from a given initial vector (with nonnegative components) by all distinct permutations and sign changes of the components of the initial vector. As for general group codes for the AWGC an important problem is to find the initial vector yielding the largest minimum Euclidean distance between two code vectors. This problem was stated by Slepian in [2] but no explicit solution has been given so far. The problem can be formulated as an integer programming problem, as is done in [4]. In [4], also, an algorithm for the calculation of optimal initial vectors is derived. We derive explicit formulas for these vectors for permutation modulation, both Variant 1 and Variant 2. The optimal initial vectors turn out to have sampled Gaussian amplitude distribution.

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The performance of optimized permutation modulation is compared to more conventional modulation forms, such as binary antipodal signaling and phase-shift-keying. The comparison turns out in favor of permutation modulation that requires 2–4 dB less power for the same minimum Euclidean distance.

II. Permutation Modulation

The initial vector is

$$x_0 = (\underbrace{\mu_1\mu_1\cdots\mu_1}_{m_1}\,\underbrace{\mu_2\cdots\mu_2}_{m_2}\cdots\cdots\underbrace{\mu_k\cdots\mu_k}_{m_k}).$$

The number of components in x_0 is

$$n=\sum_{j=1}^{k} m_j.$$

μ_i are real numbers and since the ordering is not important on the AWGC we order the components in ascending order

$$\mu_1<\mu_2<\mu_3\cdots<\mu_k.$$

The code, Variant 1, consists of all the distinct vectors obtained by permuting the components of x_0. Hence the number of codewords is

$$M_1=\frac{n!}{m_1!m_2!\cdots m_k!}.$$

In Variant 2 we also make all possible sign changes of the components in x_0. For simplicity we assume that $\mu_i \ge 0$ for all i. The number of codewords is

$$M_2=\frac{n!}{m_1!m_2!\cdots m_k!}2^h$$

where $h=n-m_1$ if $\mu_1=0$ and $h=n$ if $\mu_1>0$.

One of the nice features of permutation modulation is that maximum-likelihood detection is easy. For Variant 1 the m_1 smallest components of the received vector are associated with μ_1, the m_2 next smallest with μ_2, etc. This yields a vector with the same composition as x_0. For Variant 2 the same procedure is applied on the magnitudes of the components of the received vector. The

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components of the estimated vector then get the same signs as the components of the received vector. Slepian proved in [3] that both detectors are maximum-likelihood for the AWGC.

III. Minimum Distance and Signal Energy

An important performance measure is the minimum Euclidean distance, $d_{\min}$, between the code vectors. With a high signal-to-noise ratio a code with lower minimum Euclidean distance yields lower probability of error. For Variant 1 we assume that $m_i > 0$ for $i = 1 \cdots k$. Then the minimum distance is

$$d_{\min} = \min_{1 \le i < k} \sqrt{2}\,(\mu_{i+1} - \mu_i).$$

For Variant 2 the same is true if $\mu_1 = 0$.

If $\mu_1 > 0$ then

$$d_{\min} = \min \begin{cases} 2\mu_1 \\ \min\limits_{1 \le i < k} \sqrt{2}\,(\mu_{i+1} - \mu_i) \end{cases}.$$

To distinguish between the two cases for Variant 2 we use the following notation:

Variant 2a: $\mu_1 = 0,\ m_i > 0, \quad \text{for } i = 1, \cdots k,$

Variant 2b: $\mu_1 > 0,\ m_i > 0, \quad \text{for } i = 1, \cdots k.$

The signal energy corresponds to the squared length of x_0, i.e.,

$$E = \sum_{i=1}^{k} m_i \mu_i^2.$$

We are now interested in maximizing $d_{\min}$ with E and $\{m_i\}$ fixed, or equivalently, to minimize E with $d_{\min}$ fixed. Since E is increasing with $|\mu_1|$ it is evidently best to choose the same difference $\mu_{i+1} - \mu_i$ for all i. We now take

$$\mu_{i+1} - \mu_i = 1, \qquad 1 \le i < k,$$

i.e.,

$$d_{\min} = \sqrt{2}.$$

For Variant 2b we obtain the same minimal distance and minimal energy if

$$\mu_1 = 1/\sqrt{2},$$

which is also postulated.

IV. Optimized Permutation Modulation

Once $d_{\min}$ is fixed to $\sqrt{2}$, permutation modulation is characterized by two parameters. We have the number of codewords M and the signal energy E. We have chosen to minimize E with fixed M. Also the dimensionality n is fixed. The result is called optimized permutation modulation. It is shown in the Appendix that the optimal initial vector obeys the following relations.

Variant 1:

$$m_i = \operatorname{int} e^{-(\eta + \mu_i^2)/\lambda}$$

$$\mu_i = i - \frac{k+1}{2}.$$

Variant 2a:

$$m_1 = \operatorname{int} \frac{1}{2} e - \eta/\lambda.$$

For $i > 1$:

$$m_i = \operatorname{int} e^{-(\eta + \mu_i^2)/\lambda}$$

$$\mu_i = i - 1.$$

Variant 2b:

$$m_i = \operatorname{int} e^{-(\eta + \mu_i^2)/\lambda}$$

$$\mu_i = i - 1 + 1/\sqrt{2}.$$

In all these expressions λ and η are parameters ($\lambda > 0$ and $\eta < 0$). Each pair (λ, η) represent a particular code for optimized permutation modulation. The values of m_i are calculated from the previous relation and then the corresponding values for E and n are found. It is interesting to note that the optimal codes have quantized sampled Gaussian amplitude distribution. Such a distribution was guessed by Slepian [3, p. 233, comment 4)] to be optimal. In fact several of Slepian's codes obey these relations (for example no. 18 and 19 in his Table 1 on page 232 in [3]).

The performance of some examples of optimized permutation modulation is shown in Fig. 1. We have compared their performance with binary antipodal signals of

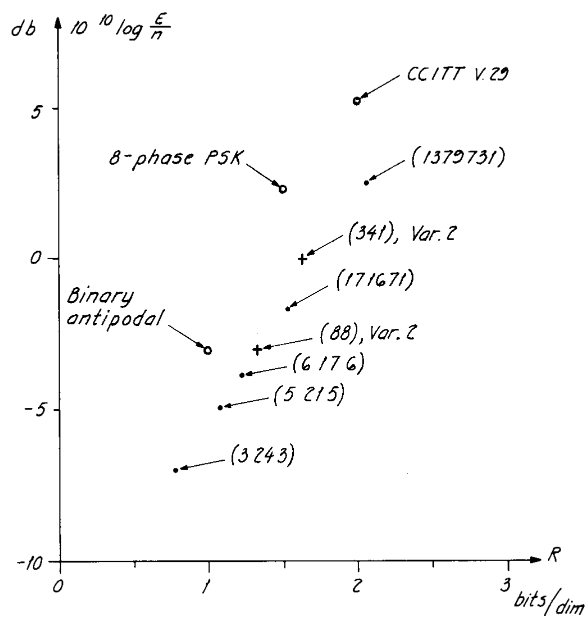

Fig. 1. Required energy (in db) as function of rate for optimized permutation modulation compared to phase- and amplitude modulation. Parameter is vector $m = (m_1, \cdots, m_k)$.

amplitude $1/\sqrt{2}$, yielding a minimum distance of $\sqrt{2}$, with 8 phase PSK and with PSK signal V.29, standardized by CCITT. All have the same minimum distance and as can be seen from Fig. 1, they require 2.5 to 4.0 dB.

V. Conclusion

Optimized permutation modulation offers better performance (2.5–4.0 dB less energy for the same minimum distance) than binary antipodal signaling or PSK. Still the maximum-likelihood detector is in principle more energy, a simple ordering operation. This does not require an amplitude reference and the detection is thus independent of the attenuation on the channel.

Appendix

Our goal here is to minimize the signal energy E with fixed number of codewords M i.e., fixed rate R and fixed dimensionality n,

$$E=\sum_{i=1}^{k} m_i\mu_i^2 \qquad n=\sum_{i=1}^{k} m_i$$

$$R=\frac{1}{n}\ln M.$$

Variant 1:

$$R_1=\frac{1}{n}\ln\frac{n!}{m_1!m_2!\cdots m_k!}$$

$$R_1=\frac{1}{n}\left[\sum_{j=1}^{n}\ln j-\sum_{i=1}^{k}\sum_{j=1}^{m_i}\ln j\right].$$

We want to minimize E, and we use a technique similar to the Lagrange multiplier. We form the function

$$f_1(m_1,\cdots,m_k)=\sum_{i=1}^{k} m_i\mu_i^2+\lambda\sum_{i=1}^{k}\sum_{j=1}^{m_i}\ln j+\eta\sum_{i=1}^{k} m_i.$$

Note that the first sum in the expression for R is a constant and is thus omitted in the function f_1.

When m_i increases, f_1 is passing through a minimum when the following inequalities are satisfied:

$$\begin{cases} f_1(m_1,\cdots,m_i+1,\cdots,m_k)-f_1(m_1,\cdots,m_i,\cdots,m_k)>0\\ f_1(m_1,\cdots,m_i-1,\cdots,m_k)-f_1(m_1,\cdots,m_i,\cdots,m_k)\geq 0.\end{cases}$$

Inserting f_1 yields

$$\begin{cases}\mu_i^2+\lambda\ln(m_i+1)+\eta>0\\ \mu_i^2+\lambda\ln m_i+\eta\leq 0.\end{cases}$$

These two inequalities can only be satisfied simultaneously if $\lambda>0$ and $\eta<0$. Dividing by λ yields

$$m_i=\operatorname{int} e-(\eta+\mu_i^2)/\lambda.$$

Here int x means "the integer part of x."

The amplitude distribution for the components of the signal vectors in optimized permutation modulation is thus a quantized Gaussian distribution. Since the signal energy (for given $m_1,\cdots,m_k$) is minimized if

$$\sum_{i=1}^{k} m_i\mu_i=0.$$

In [3, (19), p. 231)] we may choose the μ values

$$\mu_i=\cdots,-2,-1,0,1,2,\cdots,\qquad \text{for odd } k$$

$$\mu_i=\cdots,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\cdots,\qquad \text{for even } k$$

or

$$\mu_i=i-\frac{k+1}{2}.$$

The optimal values for m_i may now be inserted in the expressions for E and n and we can then solve for λ and η. Another way to proceed is to regard λ and η as parameters. Each pair (λ,η) $(\lambda>0,\ \eta<0)$ then yields an optimal set $\{m_i\}$ from which we can calculate E and R.

Variant 2a:

$$R_{2a}=\frac{1}{n}\ln\frac{n!}{m_1!\cdots m_k!}2^{n-m_1}$$

$$R_{2a}=\frac{1}{n}\left[(n-m_1)\ln 2+\sum_{j=1}^{h}\ln j-\sum_{i=1}^{k}\sum_{j=1}^{m_i}\ln j\right].$$

We form the function

$$f_{2a}(m_1,\cdots,m_k)=\sum_{i=1}^{k} m_i\mu_i^2+\lambda\left(m_1\ln 2+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\ln j\right)+\eta\sum_{i=1}^{k} m_i$$

and proceed as for Variant 1. The result is the following:

$$\begin{cases}\mu_1^2+\lambda[\ln 2+\ln(m_1+1)]+\eta>0,\\ \mu_1^2+\lambda[\ln 2+\ln m_1]+\eta\leq 0,\end{cases}\qquad \text{for } i=1$$

$$\begin{cases}\mu_i^2+\lambda\ln(m_i+1)+\eta>0,\\ \mu_i^2+\lambda\ln m_i+\eta\leq 0,\end{cases}\qquad \text{for } i>1$$

We insert $\mu_1=0$ and obtain the following expressions:

$$m_1=\operatorname{int}\frac{1}{2}e-\eta/\lambda$$

$$m_1=\operatorname{int} e^{-(\lambda+\mu_i^2)/\eta},\qquad \text{for } i>1.$$

Here $\mu_i=0,1,2,\cdots$ for $i=1,2,3,\cdots$ or $\mu_i=i-1$.

Variant 2b:

$$R_{2b}=\frac{1}{n}\ln\frac{n!}{m_1!\cdots m_k!}2^{n}$$

$$R_{2b}=\frac{1}{n}\left[n\ln 2+\sum_{j=1}^{n}\ln j-\sum_{i=1}^{k}\sum_{j=1}^{m_i}\ln j\right].$$

We form the function

$$f_{2b}(m_1,\cdots,m_k)=\sum_{i=1}^{k} m_i\mu_i^2+\lambda\sum_{i=1}^{k}\sum_{j=1}^{m_i}\ln j+\eta\sum_{i=1}^{k} m_i.$$

This is the same as f_1, and we thus obtain the same result as for Variant 1 as

$$m_i=\operatorname{int} e^{-(\eta+\mu_i^2)/\lambda}.$$

Here $\mu_i=1/\sqrt{2},1+1/\sqrt{2},2+1/\sqrt{2},\cdots$, for $i=1,2,3,\cdots$ or

$$\mu_i=i-1+\frac{1}{\sqrt{2}}.$$

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