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# On the classification of binary self-dual codes admitting imprimitive rank 3 permutation groups

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**Abstract**
One of the questions of current interest in coding theory is the following: given a finite non-solvable permutation group $G$ acting transitively on a set $\Omega$, under what conditions on $G$ are self-dual codes invariant under $G$ existent or nonexistent? In this paper, this problem is investigated under the hypothesis that the group $G$ is an imprimitive rank 3 permutation group. It is proven that if $G$ is an imprimitive rank 3 permutation group acting transitively on the coordinate positions of a self-dual binary code $C$ then $G$ is one of $\mathrm{M}_{11}$ of degree 22; $\mathrm{Aut}(\mathrm{M}_{12})$ of degree 24; $\mathrm{PSL}(2, q)$ of degree $2(q + 1)$ for $q \equiv 1 (\mathrm{mod}\, 4)$; $\mathrm{PSL}(m, q)$ of degree $2 \times \frac{q^m-1}{q-1}$ for $m \geq 3$ odd and $q$ an odd prime; $\mathrm{PSL}(m, q)$ of degree $2 \times \frac{q^m-1}{q-1}$ for $m \geq 4$ even and $q$ an odd prime, and $\mathrm{PSL}(3, 2)$ of degree 14. When combined with a result on the classification of binary self-dual codes invariant under primitive rank 3 permutation groups of almost simple type this yields a result on the non-existence of extremal binary self-dual codes invariant under quasiprimitive rank 3 permutation groups of almost simple type.

**Keywords** Imprimitive rank 3 groups · Binary self-dual codes · Automorphism groups

**Mathematics Subject Classification** 20D45 · 94B05

## 1 Introduction

It is a fundamental problem in coding theory to classify self-dual codes of moderate lengths.

An approach that is often considered in addressing the problem of whether a self-dual code $C$ of given length $n$ exists is to assume the invariance of $C$ under some

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non-trivial automorphism group and then try either to construct such a code or to prove its non-existence (see [13,14]). In order to find examples of binary self-dual codes of large length $n$ a non-trivial automorphism group might be useful.

Under the assumption that $G$ is a sporadic almost simple group, a number of self-dual codes of length $n \leq 1000$ were determined by Chigira, Harada and Kitazume in [3]. Further, the authors give a construction of a binary code $C(G, \Omega)$ as the dual of a code spanned by fixed points of involutions of a permutation group $G$ on a set $\Omega$. In fact, they showed that any binary self-dual code $\mathcal{C}$ satisfies $C(G, \Omega)^{\perp} \subseteq \mathcal{C} \subseteq C(G, \Omega)$. Recently, Mukwembi, Rodrigues and Shumba in [16] (see also [22]) extended the results of [3] to length $n \leq 4095$ and found several examples of binary self-dual codes invariant under sporadic groups of almost simple type. The said paper also examined the question of existence of extremal binary self-dual codes invariant under the prescribed type of groups, and showed that the only binary extremal self-dual codes invariant under a finite almost simple group with a sporadic socle are the extended binary Golay code admitting $\mathrm{M}_{12}$:2 in its imprimitive action of degree 24, and a $[44, 22, 8]_2$ singly-even self-dual code of length 44 admitting $\mathrm{M}_{22}$:2.

The rank of a permutation group $G$ transitive on a set $\Omega$ is the number of orbits of $G_\omega$, where $\omega$ is a point of $\Omega$, in $\Omega$. Hence, a transitive group $G$ has rank 2 on the set $\Omega$ if and only if $G$ is 2-transitive on $\Omega$. In a related study, under the assumption that $G$ acts 2-transitively on the coordinate positions of an extremal code, Malevich and Willems [14] gave a classification of extremal self-dual doubly-even codes $C$ invariant under $G$, stopping short of showing the non-existence of a putative binary extremal self-dual doubly-even code of length 1024 on which a group $\mathrm{T} \rtimes \mathrm{SL}(2, 2^5)$ acts as an automorphism group. Notice that T is an elementary abelian group of order 1024. In [4] Chigira, Harada and Kitazume completed the characterisation by showing that there does not exist a binary extremal self-dual doubly-even code of length 1024 on which $\mathrm{T} \rtimes \mathrm{SL}(2, 2^5)$ acts as a permutation group of automorphisms. Malevich and Willems, and later Chigira et al, have thus answered the question of existence of extremal binary self-dual codes when the group $G$ has rank 2.

A transitive group $G$ acting on a set $\Omega$ has rank 3 if and only if for every point $\omega$ in $\Omega$, $G_\omega$ has two orbits besides $\{\omega\}$. Rank 3 groups can be either primitive or imprimitive. Under the assumption that $G$ is a rank 3 group, it seems natural to ask: which binary self-dual codes have rank 3 permutation groups acting on them? Further, it is of interest to examine which of these binary self-dual codes (singly-even or doubly-even), if any exist, are extremal?

The above questions have been addressed for some of the different types of primitive rank 3 groups. In [20] we classified all $G$-invariant binary self-dual codes admitting $G$ a primitive rank 3 permutation group of almost simple type. Those admitting $G$ a primitive rank 3 permutation group of grid type are examined in [21]. The question of existence (respectively non-existence) of binary self-dual codes invariant under the primitive rank 3 permutation groups of affine type remains open.

In [22, Theorem 4.5] a partial statement of results related with the existence (respectively non-existence) of binary self-dual codes invariant under an imprimitive rank 3 permutation group $G$ was given. In this paper we complete and improve on the results given in [22, Theorem 4.5]. In doing so we determine up to equivalence all $G$-invariant binary self-dual codes on which $G$ acts transitively on the coordinate positions, where

$G$ is a prescribed finite imprimitive rank 3 permutation group. The other purposes of the paper are to examine whether these binary self-dual codes invariant under $G$ are doubly-even (respectively singly-even) and to investigate if any of them are extremal.

As a consequence we prove the following main results.

**Theorem 1.1** *Let $G \leq S_n$ be an imprimitive rank 3 permutation group. Then there exists a self-dual code $C \leq \mathbb{F}_2^n$ with $G \leq \mathrm{Aut}(C)$ if and only if $G$ and $C$ are as follows:*

(a) $G \cong \mathrm{M}_{11}$ *of degree* 22 *and* $C = [22, 11, 2]_2$.
(b) $G \cong \mathrm{Aut}(\mathrm{M}_{12})$ *of degree* 24 *and* $C = [24, 12, 8]_2$.
(c) $G \cong \mathrm{PSL}(2, q)$ *of degree* $2(q + 1)$ *and* $C = [2(q + 1), q + 1, 2]_2$ *with* $q = p^{2t} \equiv 1 \pmod 4$, $t \geq 1$, $p \equiv 3 \pmod 4$ *and* $p$ *a prime.*
(d) $G \cong \mathrm{PSL}(m, q)$ *of degree* $2 \times \frac{q^m-1}{q-1}$ *and*

$$C = \begin{cases} [2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 2]_2, & \text{for } m \geq 3 \text{ odd and } q \text{ an odd prime;} \\ [2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 2]_2, & \text{for } m \geq 4 \text{ even, } q = p^2 \equiv 1 \pmod 4 \text{ and } p \text{ a prime;} \\ [2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 4]_2, & \text{for } m \geq 4 \text{ even, } q = p^2 \equiv 1 \pmod 4 \text{ and } p \text{ a prime;} \\ [2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 4]_2, & \text{for } m \geq 4 \text{ even and } q \equiv 1 \pmod 4; \\ [2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 2]_2, & \text{for } m \geq 4 \text{ even and } q \equiv 1 \pmod 4. \end{cases}$$

(e) $G \cong \mathrm{PSL}(3, 2)$ *of degree* 14 *and* $C$ *is a* $[14, 7, 2]_2$ *code.*

**Theorem 1.2** *Let $G \leq S_n$ be an imprimitive rank 3 permutation group. Then $C \leq \mathbb{F}_2^n$ with $G \leq \mathrm{Aut}(C)$ is a binary self-dual doubly-even code of length $n$ if and only if the following holds:*

(a) $G \cong \mathrm{Aut}(\mathrm{M}_{12})$ *and* $C = [24, 12, 8]_2$ *is isomorphic to the extended binary Golay code.*
(b) $G \cong \mathrm{PSL}(m, q)$ *and* $C = [2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 4]_2$ *where* $m \geq 4$ *is even,* $q = p^2 \equiv 1 \pmod 4$ *and* $p$ *a prime.*

**Theorem 1.3** *Let $C$ be an extremal doubly-even binary self-dual code admitting an imprimitive rank 3 permutation automorphism group $G$. Then $C$ is isomorphic to the extended binary Golay code and $G$ is isomorphic to* $\mathrm{Aut}(\mathrm{M}_{12})$.

The paper is organized as follows: in Sect. 2 we give a brief description of the terminology and background to be used in the paper; in Sect. 3 following a series of propositions we give the proof of Theorem 1.1. The proof for the individual examples of imprimitive rank 3 groups as well as that for the infinite families of type PSL follows by a case-by-case analysis in the same section. In Sect. 4 we prove Theorem 1.2. Finally, in Sect. 5 we discuss the question of existence (respectively non-existence) of extremal binary self-dual codes admitting an imprimitive rank 3 group and prove Theorem 1.3.

## 2 Background and definitions

In this section, we state some useful facts in coding theory and finite group theory. The notation for the structure of groups is as given in the ATLAS [6]. We denote by $C_p$ the cyclic group of order $p$.

Throughout the paper we assume $\kappa = \mathbb{F}_2$, and $G$ is a finite permutation group acting on a finite non-empty set $\Omega$, i.e. there is a $G$-action on $\Omega$, namely, a map $\cdot : G \times \Omega \longrightarrow \Omega$ given by $(g, x) \mapsto g \cdot x$, satisfying $(g \cdot h) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and all $x \in \Omega$, and that $1 \cdot x = x$ for all $x \in \Omega$.

Then $\kappa\Omega = \{\sum_{x\in\Omega} g_x x \mid g_x \in \kappa\}$ is a vector space over $\kappa$ with basis $\Omega$. Extending the $G$-action on $\Omega$ linearly, $\kappa\Omega$ becomes a $\kappa G$-module, called a $\kappa G$-permutation module with permutation basis $\Omega$.

The $\kappa$-vector space $\kappa\Omega$ is equipped with a non-degenerate symmetric bilinear form

$$\left\langle \sum_{x\in\Omega} g_x x, \sum_{x\in\Omega} h_x x \right\rangle = \sum_{x\in\Omega} g_x h_x, \ \forall \mathbf{g} = \sum_{x\in\Omega} g_x x \text{ and } \mathbf{h} = \sum_{x\in\Omega} h_x x \in \kappa\Omega$$

called the standard inner product on $\kappa\Omega$. For any $a \in G$ and any $\mathbf{g}$ and $\mathbf{h}$ as given above, we have

$$\begin{aligned} \langle a(\mathbf{g}), a(\mathbf{h})\rangle &= \left\langle a\left(\sum_{x\in\Omega} g_x x\right), a\left(\sum_{x\in\Omega} h_x x\right)\right\rangle \\ &= \left\langle \sum_{x\in\Omega} g_x ax, \sum_{x\in\Omega} h_x ax \right\rangle = \sum_{x\in\Omega} g_x h_x \\ &= \langle \mathbf{g}, \mathbf{h}\rangle. \end{aligned}$$

So, the standard inner product on the vector space $\kappa\Omega$ is $G$-invariant in the following sense:

$$\langle a(\mathbf{g}), a(\mathbf{h})\rangle = \langle \mathbf{g}, \mathbf{h}\rangle, \ \ \forall a \in G, \forall \mathbf{g}, \mathbf{h} \in \kappa\Omega.$$

We define the dual code $C^\perp$ by $C^\perp = \{v \in \kappa^n \mid \langle v, c\rangle = 0, \text{ for all } c \in C\}$. If $C \subseteq C^\perp$ we call $C$ self-orthogonal. If $C = C^\perp$ we say that the code $C$ is self-dual.

For $U$ a right $G$-module its dual module $U^* = \mathrm{Hom}_{\mathbb{F}_2}(U, \mathbb{F}_2)$ is a right $G$-module via $(fg)(u) = f(ug^{-1}),$ for $f \in U^*, g \in G,$ and $u \in U$. If $U \cong U^*$ then $U$ is said to be self-dual.

If $C$ is a $\kappa G$-submodule of $\kappa\Omega$, then for any $a \in G$ and $\mathbf{u}' \in C^\perp$, and for any $\mathbf{u} \in C$, by the $G$-invariance of the inner product we have

$$\langle a\mathbf{u}', \mathbf{u}\rangle = \langle a\mathbf{u}', aa^{-1}\mathbf{u}\rangle = \langle \mathbf{u}', a^{-1}\mathbf{u}\rangle = 0,$$

so $a\mathbf{u}' \in C^\perp$, i.e., $C^\perp$ is $G$-invariant. Hence, $C^\perp$ is a $\kappa G$-submodule.

Two linear codes are *isomorphic* if they can be obtained from one another by permuting the coordinate positions. For a linear code $C$ of length $n$ over $\kappa$, a permutation

of the components of a codeword of length $n$ is said to be a permutation automorphism of $C$ if the permutation maps codewords to codewords. By $\mathrm{Aut}(C)$ we denote the automorphism group of $C$ consisting of all the permutation automorphisms of $C$. With this we have that $G$ acts on $C$ and thus $G \leq \mathrm{Aut}(C)$ so that the code $C$ becomes a $\kappa G$-submodule of the permutation module $\kappa\Omega$, and so is $C^*$.

Clearly, the two duals $C^*$ and $C^\perp$ are not the same object. However, there is a connection between these two notions of duality, i.e., if $C$ is a code of length $n$ over $\kappa$ and $G \leq \mathrm{Aut}(C)$, then $C^* \cong \kappa^n/C^\perp$ (as $\kappa G$-modules, in particular, as vector spaces). Naturally, for self-dual codes we have $C^* \cong \kappa^n/C$.

A binary code $C$ is *doubly-even* if all codewords of $C$ have weight divisible by 4. The weight distribution of a code $C$ is the sequence $\{A_i | i = 0, 1, \ldots, n\}$, where $A_i$ is the number of codewords of weight $i$. The polynomial $W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$ is called the weight enumerator of $C$. The weight enumerator of a code $C$ and its dual $C^\perp$ are related via MacWilliams identity.

In an attempt to make the paper self-contained we collect some relevant facts on the finite rank 3 groups. The study of rank 3 permutation groups dates back to the paper [12] of Donald Higman. Rank 3 groups can be either primitive or imprimitive.

A subset $B$ of $\Omega$ is a block for $G$ if for all $g \in G$ either

$$B^g = B \text{ or } B^g \cap B = \varnothing.$$

If $G$ is a permutation group on a set $\Omega$, then a partition $\mathcal{B}$ of $\Omega$ is said to be $G$-invariant if the elements of $G$ permute the blocks of $\mathcal{B}$ blockwise, i.e., for $B \in \mathcal{B}$ and $g \in G$, the set $B^g$ is also a block of $\mathcal{B}$. The blocks of a $G$-invariant partition are called blocks of imprimitivity for $G$. If $G$ is transitive on $\Omega$ then all blocks of a $G$-invariant partition $\mathcal{B}$ of $\Omega$ have the same cardinality and $G$ acts transitively on $\mathcal{B}$. When $G$ is transitive, a $G$-invariant partition of $\Omega$ is called a system of imprimitivity or a block system for $G$. Furthermore, every permutation group $G$ on $\Omega$ preserves the two partitions $\{\Omega\}$ and $\{\{\alpha\} \mid \alpha \in \Omega\}$; these are called trivial partitions of $\Omega$, and their blocks, $\Omega$ and $\{\alpha\}$ for $\alpha \in \Omega$, are called trivial blocks or trivial systems of imprimitivity. All other partitions of $\Omega$ are called non-trivial. A permutation group $G$ is said to be primitive on $\Omega$ if $G$ is transitive on $\Omega$ and the only $G$-invariant partitions of $\Omega$ are the trivial ones. Also $G$ is said to be imprimitive on $\Omega$ if $G$ is transitive on $\Omega$ and $G$ preserves some non-trivial partition of $\Omega$.

A permutation group is called quasiprimitive if every non-trivial normal subgroup is transitive. All quasiprimitive permutation groups of rank 3 are known. They are either primitive or imprimitive and almost simple, see [7, Corollary 1.3].

If $G$ is a primitive rank 3 permutation group of finite degree $n$ then one of the following holds:

(a) *Almost simple type*: $S \trianglelefteq G \leq \mathrm{Aut}(S)$, where $S = \mathrm{soc}(G)$ is a nonabelian simple group;
(b) *Grid type:* $S \times S \trianglelefteq G \leq S_0 \wr Z_2$, where $S_0$ is a 2-transitive group of degree $n_0$, with $S \trianglelefteq S_0 \leq \mathrm{Aut}(S)$, $S$ nonabelian simple, and $n = {n_0}^2$;
(c) *Affine type:* $G = SG_0$, where $S$ is an elementary abelian $p$-group acting regularly on a vector space $V$, $G_0$ is an irreducible subgroup of $\mathrm{GL}(m, p)$ and $G_0$ has exactly 2 orbits on the nonzero vectors of $V$.

A primitive rank 3 group $G$ has a unique minimal normal subgroup $S$, called its socle, and $S$ can be a non-abelian simple group, a direct product of two isomorphic non-abelian simple groups, or elementary abelian. When $S$ is elementary abelian, $G$ is said to be of affine type; and when $S$ is a direct product of two non-abelian simple groups, $G$ is said to be of product action type.

In what follows we give a brief but complete overview of what is known on the classification of finite imprimitive rank 3 permutation groups. The relevant details and results related with the said classification can be found in [7] from where most of the material was drawn.

Suppose that $G$ is a transitive imprimitive permutation group acting on a set $\Omega$, and suppose that $G$ preserves a non-trivial block-system $\mathcal{B}$ on $\Omega$. Referring to the Embedding Theorem for imprimitive groups (see [1]) in [7] the authors consider a block, say $B \in \mathcal{B}$ and identify $\Omega$ with the set $B \times \{1, \ldots, n\}$, where $n = |\mathcal{B}|$ so that $G$ is viewed as a subgroup of the wreath product $K \wr Y$ where $G^{\mathcal{B}} \cong Y \leq S_n$ and $K := G_B^B$ is the component of $G$. Here $G^{\mathcal{B}}$, the subgroup of the symmetric group $S_{\mathcal{B}}$ induced by $G$ is transitive and $G_B^B$ the subgroup of the symmetric group $S_B$ induced by the setwise stabiliser $G_B$ is also transitive. The partition $\mathcal{B}$ is identified with $\{B \times \{i\} \mid i \in \{1, \ldots, n\}\}$. In this way, if $G$ has rank 3, then $G_B^B$ and $G^{\mathcal{B}}$ are 2-transitive and $\mathcal{B}$ is the unique system of imprimitivity, and conversely if $K \leq S_B$ and $Y \leq S_n$ are both 2-transitive then $K \wr Y$ has rank 3.

In essence, there are two infinite families of imprimitive rank 3 groups, namely $\mathrm{PSL}(2, q)$ and $\mathrm{PSL}(m, q)$ with some additional conditions, and a finite number of individual imprimitive examples. Below we collect the pertinent results from [7]:

**Result 1** (Devillers et al. [7, Theorem 1.1]) *Suppose $G$ is an imprimitive group acting on a set $\Omega = B \times \{1, \ldots, n\}$ with*

(i) *$G_B^B$ a 2-transitive almost simple group with socle $S$;*
(ii) *$G^{\mathcal{B}} \leq S_n$ a 2-transitive group.*

*Then $G$ has rank 3 if and only if one of the following holds:*

(1) *$S^n \leq G$;*
(2) *$G$ is quasiprimitive and rank 3 on $\Omega$;*
(3) *$n = 2$ and $G = \mathrm{M}_{10}$, $\mathrm{PGL}(2, 9)$ or $\mathrm{Aut}(A_6)$ acting on 12 points;*
(4) *$n = 2$ and $G = \mathrm{Aut}(\mathrm{M}_{12})$ acting on 24 points.*

We note the following result which is given in general for quasiprimitive imprimitive rank 3 permutation groups but we state it for the restricted class of quasiprimitive imprimitive rank 3 permutation groups of even degree, since our interest is to examine binary self-dual codes and these must be of even length. Recall that we take the length of the code to be the degree of the imprimitive permutation representation.

**Result 2** (Devillers et al. [7, Theorem 1.2]) *Let $G$ be a transitive imprimitive permutation group of rank 3 acting on a set $\Omega$. Let $n$ be the number of blocks and $m$ be the size of the blocks. Then $G$ is quasiprimitive of even degree if and only if $G$, $n = |\mathcal{B}|$, $m = |B|$ and $G_B^B$ are in one of the lines of Table 1.*

By Result 2 the quasiprimitive and imprimitive rank 3 groups of almost simple type that occur with even degree are listed in Table 1.

**Table 1** Quasiprimitive imprimitive rank 3 groups that occur with even degree $|B||\mathcal{B}|$

| Line | $G$ | $\lvert\mathcal{B}\rvert$ | $\lvert B\rvert$ | $G_B^B$ | Extra conditions |
|---|---|---|---|---|---|
| 1 | $M_{11}$ | 11 | 2 | $C_2$ | |
| 2 | $G \trianglerighteq \mathrm{PSL}(2,q)$ | $q+1$ | 2 | $C_2$ | $q = p^f \geq 4$, $f \geq 1$, (i) $q \equiv 1 (\mathrm{mod}\, 4)$, (ii) $G = \langle \mathrm{PSL}(2,q), \alpha^i\delta\rangle$ where $i$ divides $f$, $f/i$ is even, and either $p^i \equiv 3 (\mathrm{mod}\, 4)$, or $p^i \equiv 1 (\mathrm{mod}\, 4)$ and $f/i \equiv 0 (\mathrm{mod}\, 4)$ (iii) $G_{(B)} = Q \rtimes \langle \delta^4, \alpha^i\delta^t\rangle$ with $t = 1$ or 3 |
| 3 | $G \trianglerighteq \mathrm{PSL}(m,q)$ | $\frac{q^m-1}{q-1}$ | $s$ | $\mathrm{AGL}(1,s)$ | $q = p^f$, $f \geq 1, m \geq 3$, $s$ prime , $ord(p^i \bmod s) = s-1$, $ds \mid (q-1)$, $ds \mid (r+\lambda d)\frac{q-1}{p^i-1}$ for some $\lambda \in [0, s-1]$, where $d \mid r\frac{(q-1)}{(p^i-1)}$, and $(sd, s) = d$. See additional conditions in Remark 2(b) |
| 4 | $\mathrm{PGL}(3,4)$ | 21 | 6 | $\mathrm{PSL}(2,5)$ | |
| 5 | $\mathrm{P\Gamma L}(3,4)$ | 21 | 6 | $\mathrm{PGL}(2,5)$ | |
| 6 | $\mathrm{PSL}(5,2)$ | 31 | 8 | $A_8$ | |
| 7 | $\mathrm{P\Gamma L}(3,8)$ | 73 | 28 | $Ree(3)$ | |
| 8 | $\mathrm{PSL}(3,2)$ | 7 | 2 | $C_2$ | |

The following two remarks are relevant for Line 2 and Line 3 of Table 1

***Remark 1*** ([7, Remark 6]) The following details are relevant for the three cases given in Line 2 of Table 1. Let $S = \mathrm{PSL}(2, q) \leq G \leq \mathrm{P0L}(2, q) = A$ with $A$ acting on $\mathcal{B}$ of size $q + 1$ where $q = p^f \geq 4$ with $p$ a prime and $f \geq 1$, and let $B \in \mathcal{B}$. Then $G_{(B)} = ((Q \rtimes \langle\delta\rangle) \rtimes \langle\alpha\rangle) \cap G$ where $Q$ is an elementary abelian group of order $q$, $\delta$ of order $q - 1$ and $\alpha$ is of order $f$ and $G_{(B)}$ denotes the setwise stabilizer of $B$. Let $i = |G/G \cap \mathrm{PGL}(2, q)|$ then $i$ divides $f$.

***Remark 2*** According to [7, Theorem 1.4] a group $G$ in Line 2 and Line 3 of Table 1 is almost simple and block faithful if and only if the following conditions on $G$, $n = |\mathcal{B}|$ and $|B|$ are satisfied:

(a) Line 2: $G \trianglerighteq \mathrm{PSL}(2, q)$, $n = q + 1$, $|B| = 2$ and $q \equiv 1 \pmod 4$, or $q \equiv 3 \pmod 4$ and $G \geqslant \mathrm{PGL}(2, q)$ or $i = |G/(G \cap \mathrm{PGL}(2, q))|$ is even.

(b) Line 3: $G \trianglerighteq \mathrm{PSL}(m, q)$, $n = \frac{q^m-1}{q-1}$, $|B| = s$ and $s$ is prime, $ord(p^i \bmod s) = s - 1$, $ds|(q - 1)$, $ds|(r + \lambda d)\frac{q-1}{p^i-1}$ for some $\lambda \in [0, s - 1]$, where $d|r\frac{(q-1)}{(p^i-1)}$, and $gcd(sd, s) = d$. See Sect. 3.2 for explanations on $d$, $r$ and $i$.

For the convenience of the reader, we list results concerning the existence of binary self-dual codes invariant under permutation groups which will be used frequently throughout the paper.

**Result 3** (Günther and Nebe. [11, Theorem 2.1]) *Let $G \leq S_n$. Then there exists a self-dual code $C \subseteq \mathbb{F}_2^n$ with $G \leq \mathrm{Aut}(C)$ if and only if every self-dual simple $\mathbb{F}_2G$-module $U$ occurs in the $\mathbb{F}_2G$-module $\mathbb{F}_2^n$ with even multiplicity.*

The next result deals with the existence of binary self-dual doubly-even codes invariant under permutation groups.

**Result 4** (Günther and Nebe. [11, Theorem 5.2]) *Let $G \leq S_n$. Then there is a self-dual doubly-even code $C = C^\perp \subseteq \mathbb{F}_2^n$ with $G \leq \mathrm{Aut}(C)$ if and only if the following three conditions are fulfilled:*

(i) $8 \mid n$;

(ii) *every self-dual composition factor of the $\mathbb{F}_2G$-module $\mathbb{F}_2^n$ occurs with even multiplicity;*

(iii) $G \leq A_n$.

### 2.1 Outline of the proof of the theorems

In order to determine whether a group $G$ given in Result 1 and in Table 1 occurs as an automorphism group of a binary self-dual code $C$ of length $n$ we exploit the structure of $C$ and its ambient space $\mathbb{F}_2^n$ as a $\mathbb{F}_2G$-module. Notice that in Result 1 the length $n$ of the code is the degree of $G$ while in Table 1 the length $n = |B||\mathcal{B}|$. In particular, note that $C \leq \mathbb{F}_2{}^n$ is of dimension $\frac{n}{2}$. Moreover, for each group $G$ in the above-mentioned list we construct all $\frac{n}{2}$-dimensional $\kappa G$-submodules $C$ and verify whether $C$ is self-dual as a code. This is done with MAGMA [2] using Result 3. We then apply Result 4 to those codes $C$ which are self-dual to verify whether they are doubly-even.

## 3 Proof of the theorems

We prove the theorem by a series of propositions considering the permutation modules associated with the imprimitive rank 3 action of $G$ as described above. We start by considering $G$ as in Result 1.

### 3.1 $G$ an imprimitive rank 3 group

Let $G$ be as in part (1) of Result 1. Then, it can be inferred from the proof of Theorem 1.1 of [7, p. 656–657] that if $G$ is block-faithful then $G$ is quasiprimitive. This implies that $G$ is as in part (2) of Result 1. Otherwise, if $G$ is not quasiprimitive and $S^n \not\leq G$ then $G$ is as in parts (3) and (4), respectively of Result 1 and the proof follows below.

Next, let $G$ be as in part (2) of Result 1. Here $G$ is a quasiprimitive group, and so $G$ is either primitive or imprimitive and almost simple. The binary self-dual codes of length $n$ admitting $G$ a primitive rank 3 group of almost simple type have been examined in [20]. Those admitting $G$ a primitive rank 3 group of grid type are addressed in [21]. The classification of binary self-dual codes invariant under a primitive rank 3 group of affine type remains open. The codes admitting a quasiprimitive imprimitive rank 3 permutation group of almost simple type will be examined in Sect. 3.2 (the possible groups under which they are invariant are those described in Table 1).

Let $G$ be as in part (3). Then $G \cong \mathrm{M}_{10}$, $\mathrm{PGL}(2, 9)$ or $\mathrm{Aut}(\mathrm{A}_6)$ in their imprimitive rank 3 representation of degree 12. For $G \cong \mathrm{M}_{10}, \mathrm{PGL}(2, 9)$ the action is that of $G$ on the cosets of a subgroup isomorphic to the alternating group $\mathrm{A}_5$ while for $G \cong \mathrm{Aut}(\mathrm{A}_6)$, $G$ acts on the cosets of a subgroup isomorphic to the symmetric group $\mathrm{S}_5$. For any choice of $G$ we have an imprimitive rank 3 representation with subdegrees 1, 5 and 6 respectively. Computations with MAGMA show that in all cases there are no $G$-invariant submodules of dimension 6.

In ending this section, let $G$ be as in part (4). Then $n = 2$ and $G = \mathrm{Aut}(\mathrm{M}_{12})$ acting on 24 points. An examination of the maximal subgroups of $\mathrm{M}_{12}$, see $\mathbb{ATLAS}$ [6, p. 33] or [17, Section 4.4.2] shows that there exists $H \leq \mathrm{M}_{12}$ such that $H \cong \mathrm{M}_{11}$. Also, there is just one $G$-conjugacy class of such subgroups. Moreover, $H$ is transitive by conjugation on the 12 subgroups $\mathrm{M}_{11}$ which form one of the two $\mathrm{M}_{12}$-conjugacy classes of such subgroups. Furthermore, this is an imprimitive rank 3 representation, with subdegrees 1, 11, 12. Through computations with MAGMA we constructed the associated permutation module $\mathbb{F}_2\Omega$ of degree 24, obtaining in it only one submodule of dimension 12. Since this submodule satisfies Result 3, it is a binary self-dual code. Let $C(\mathrm{M}_{12}{:}2, 24)$ denote this self-dual code. Since the minimum distance of $C(\mathrm{M}_{12}{:}2, 24)$ equals 8, then $C(\mathrm{M}_{12}{:}2, 24)$ is a $[24, 12, 8]_2$-code. Now, a binary self-dual code with the same parameters as those of $C(\mathrm{M}_{12}{:}2, 24)$ and invariant under a subgroup of $\mathrm{M}_{24}$ must be isomorphic to the extended binary Golay code. By the uniqueness of the extended binary Golay code, we establish the following result:

**Proposition 3.1** *Let $G = \mathrm{M}_{12}{:}2$ in its imprimitive rank 3 action on the cosets of $\mathrm{M}_{11}$. Let $\mathbb{F}_2\Omega$ be the permutation module of $G$ of dimension 24 and $C(\mathrm{M}_{12}{:}2, 24) \subset \mathbb{F}_2\Omega$ denote a submodule of dimension 12. Then $C(\mathrm{M}_{12}{:}2, 24)$ is a self-dual code of length*

24 *which admits G as permutation group of automorphisms. Further,* $C(\mathrm{M}_{12}{:}2, 24)$ *is isomorphic to the extended binary Golay code.*

Notice that Proposition 3.1 also follows by [3, Example 2.5].

### 3.2 *G* quasiprimitive imprimitive rank 3 of almost simple type

In this section we examine the existence of binary self-dual codes admitting quasiprimitive imprimitive rank 3 permutation groups of almost simple type. As noted earlier if these codes exist they will be invariant under the groups listed in Table 1.
We start by considering $G$ as in Line 1 of Table 1. Then $G \cong \mathrm{M}_{11}$ in its imprimitive representation of degree 22. It follows from [17, Section 4.2.2, p. 34] that $G$ acts rank 3 with subdegrees 1, 1, 20. The point stabilizer is a subgroup isomorphic to $\mathrm{M}_{10}$. Using MAGMA we determined the 2-modular structure of the $\mathbb{F}_2G$-permutation module of degree 22 and found a unique submodule of dimension 11. Since this submodule satisfies Result 3, it is a self-dual code over $\mathbb{F}_2$. Thus we have

**Proposition 3.2** *Let* $G = \mathrm{M}_{11}$ *in its imprimitive rank* 3 *action on the cosets of* $\mathrm{M}_{10}$. *Let* $C_2(\mathrm{M}_{11}, 22)$ *be the submodule of dimension* 11 *of the permutation module* $\mathbb{F}_2G$ *of degree* 22 *invariant under G. Then* $C_2(\mathrm{M}_{11}, 22) = [22, 11, 2]_2$ *is the unique self-dual code of length* 22 *invariant under G. Moreover,* $\mathrm{Aut}(C_2(\mathrm{M}_{11}, 22)) \cong 2 \wr S_{11} = 2^{11}{:}S_{11}$.

***Proof*** The uniqueness of the code $C_2(\mathrm{M}_{11}, 22)$ could be shown using [3, Lemma 2.3]. To show that the structure of the automorphism is as claimed we observe that $G$ acts primitively on $11 = \frac{22}{2}$ points, so that $\mathrm{Aut}(C(G, 11)) = S_{11}$. Hence applying [22, Theorem 3.1 (ii), part (a)] we deduce that $\mathrm{Aut}(C_2(\mathrm{M}_{11}, 22)) \cong 2 \wr S_{11}$. □

The next results will deal with the infinite families of quasiprimitive imprimitive rank 3 permutation groups. For this consider $G$ as in Line 2 of Table 1. Then $G$ is an almost simple group with socle $\mathrm{PSL}(2, q)$.

We give a brief overview of the projective groups $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P\Gamma L}(2, q)$, referring the reader to [9] for the definitions of $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$. Since we are dealing with finite classical groups we assume that the field of the group is a finite field of order $q$ where $q = p^f \geq 4$ and $p$ is a prime and $f \geq 1$. We denote the automorphism group of $\mathrm{PSL}(2, q)$ by $\mathrm{P\Gamma L}(2, q)$. It is obtained by adjoining field automorphisms to the transformations of $\mathrm{PGL}(2, q)$. Adjoining the field automorphisms to $\mathrm{PSL}(2, q)$ yields a subgroup of $\mathrm{P\Gamma L}(2, q)$ denoted by $\mathrm{P\Sigma L}(2, q)$ and $\mathrm{P\Sigma L}(2, q) = \mathrm{PSL}(2, q) \rtimes \mathrm{Gal}(\mathbb{F}_q)$. Moreover $|\mathrm{PSL}(2, q)| = \frac{q(q^2-1)}{\gcd(2,q-1)}$, $|\mathrm{PGL}(2, q)| = q(q^2 - 1)$, $|\mathrm{P\Gamma L}(2, q)| = fq(q^2 - 1)$ and $|\mathrm{P\Sigma L}(2, q)| = \frac{fq(q^2-1)}{\gcd(2,q-1)}$. It is a well-known fact that $\mathrm{P\Gamma L}(2, q)/\mathrm{PSL}(2, q) \cong C_f$ when $p = 2$ and $\mathrm{P\Gamma L}(2, q)/\mathrm{PSL}(2, q) \cong C_f \times C_2$ when $p \neq 2$.

Here, we use [7, Proposition 4.10] as an aid in the description of the examples of rank 3 groups with socle $\mathrm{PSL}(2, q)$. Note that these groups are of almost simple type, i.e,

$$\mathrm{PSL}(2, q) \trianglelefteq G \leq \mathrm{Aut}(\mathrm{PSL}(2, q)) = \mathrm{P\Gamma L}(2, q),$$

and satisfy the conditions stated in part (a) of Remark 2. Moreover, $G$ has degree $2(q+1)$, with a block stabilizer isomorphic to the cyclic group $C_2$. By [7, Proposition 4.10] we have that an almost simple group $G$ listed in Line 2 of Table 1 acts rank 3 if and only if the follwing three conditions hold simultaneously: (1): $q \equiv 1 (\mathrm{mod}\, 4)$, (2): $G = \langle \mathrm{PSL}(2,q), \alpha^i \delta \rangle$ where $i$ divides $f$, $f/i$ is even and either $p^i \equiv 3 (\mathrm{mod}\, 4)$, or $p^i \equiv 1 (\mathrm{mod}\, 4)$ and $f/i \equiv 0 (\mathrm{mod}\, 4)$, (3): $G_{(B)} = Q \rtimes \langle \delta^4, \alpha^i \delta^t \rangle$ with $t = 1$ or $3$, where $i = |G/(G \cap \mathrm{PGL}(2,q))|$ is even. These consitions are described in Remark 2(a). Furthermore, the two actions for $t = 1$ or 3 are not isomorphic.

Note that conditions (1), (2), and (3) given above and the fact that $G$ strictly contains $\mathrm{PSL}(2,q)$ exclude the possibility of $G = \mathrm{PSL}(2,q)$, $\mathrm{PGL}(2,q)$, $\mathrm{P\Sigma L}(2,q)$ or $\mathrm{P\Gamma L}(2,q)$ being a rank 3 group of degree $2(q+1)$. Consider $f = 2l$ for an integer $l \geq 1$. By condition (2), $i | 2l$ and since $2l/i$ is even we must have $i = 1$ or $i = l$. Direct calculations show that the examples of rank 3 quasiprimitive imprimitive $G$ with socle $\mathrm{PSL}(2,q)$ of degree $2(q+1)$ are $G = \mathrm{PSL}(2,q) \cdot C_{2l}$, the non-split extension of $\mathrm{PSL}(2,q)$ by $C_{2l}$ where $q = p^{2l}$ for some $l \geq 1$ and $p \equiv 3 (\mathrm{mod}\, 4)$. A description of these groups can be found in [5, Section 3.2].

Recall that $G$ has two non-isomorphic rank 3 actions of degree $2(q+1)$ with subdegrees 1, 1, and $2q^{2l}$ for $l \geq 1$. These rank 3 actions give rise to two inequivalent imprimitive rank 3 permutation modules of dimension $2(q+1)$ each of which containing a unique submodule of dimension $q+1$.

In Fig. 2 below we depict the diagram of the submodule structure (the composition factors can be derived from this) of the permutation module $\mathbb{F}_2\Omega$ associated to one of the two imprimitive rank 3 permutation representations of $\mathrm{PSL}(2,q) \cdot C_{2l}$ of degree $2(q+1)$. The other imprimitive rank 3 representation can be described in a similar way. Notice that the vector space dimension is given in parentheses.

Let $\mathcal{P}_{q+1} = \mathbb{F}_2\Omega_{q+1}$ denote the permutation module of dimension $q+1$. It is well-known that $G$ acts 2-transitively on $\Omega_{q+1}$. Consequently the dimension of the $\mathrm{End}_{\mathbb{F}_2 G}(\mathcal{P}_{q+1}) = 2$. A basis for $\mathrm{End}_{\mathbb{F}_2 G}$ is given by

$$\varepsilon_1 = \mathrm{id}_{(\mathcal{P}_{q+1})};$$
$$\varepsilon_2 = \left( u \mapsto \sum_{v \in \Omega_{q+1}} v \text{ for all } u \right).$$

The permutation module has structure as given in Fig. 1.

Now, let $\mathcal{P}_{2(q+1)} = \mathbb{F}_2\Omega_{2(q+1)}$ denote the permutation module for $G$ of degree $2(q+1)$. Then by Result 2 and Table 1 we have that this representation has $q+1$ blocks of imprimitivity of length 2 which are permuted 2-transitively. We establish a natural relation between $\mathcal{P}_{q+1}$ and $\mathcal{P}_{2(q+1)}$ in the following way:

$$\varphi : \mathbb{F}_2\Omega_{2(q+1)} \longrightarrow \mathbb{F}_2\Omega_{q+1}, \text{ given by } u \mapsto u + u' \tag{1}$$

where $\{u, u'\}$ is the block containing $u$.

We see that $\varphi$ is an $\mathbb{F}_2 G$ endomorphism of $\mathcal{P}_{2(q+1)}$ and $\mathrm{Im}(\varphi) \cong \mathcal{P}_{q+1}$ by identification of blocks. By the isomorphism theorem for modules we observe that

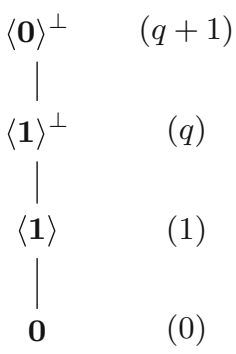


**Fig. 1** Submodule lattice for the $\mathbb{F}_2\mathrm{PSL}(2, q)$-module of dimension $(q + 1)$

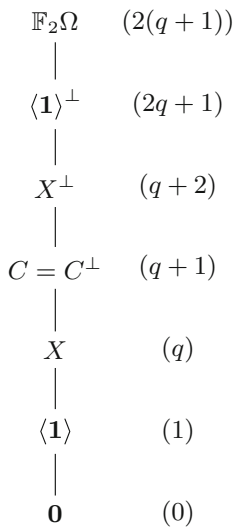


**Fig. 2** Submodule structure of the imprimitive rank 3 permutation module of $\mathbb{F}_2\mathrm{PSL}(2, q) \cdot C_{2l}$ of dimension $2(q + 1)$

$\mathcal{P}_{2(q+1)}/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi) \cong \mathcal{P}_{q+1}$. Thus $\mathrm{Ker}(\varphi) = \mathrm{Im}(\varphi)$ and so we obtain a composition series from which we obtain the complete lattice of submodules as depicted in Fig. 2. Note that in Fig. 2 the vector space dimension is given in parentheses.

Let $X = C(G, \Omega)$ denote the code of dimension $q$ and $X^\perp$ its dual code of dimension $q+2$. Then, it follows from Fig. 2 that between $X$ and $X^\perp$ are the submodules: $X$, $C = C^\perp$ and $X^\perp$, with $\dim(X) = q$ and $\dim(C) = \dim(C^\perp) = q+1$. Further, notice that the composition factors are $\kappa$ (4 times) and $L \cong \langle \mathbf{1} \rangle^\perp / X^\perp \cong X/\langle \mathbf{1} \rangle$ (dually).

We deduce the following

**Proposition 3.3** *The submodules given in Fig. 2 are all the $\mathbb{F}_2G$-submodules of $\mathbb{F}_2\Omega_{2(q+1)}$.*

Notice from Fig. 2 that there is only one submodule of dimension $q + 1$ invariant under $G$. Applying Result 3 we obtain that this module is a binary self-dual code. In the next result we state the pertinent properties of the code.

**Proposition 3.4** *Let $G = \mathrm{PSL}(2, q) \cdot C_{2l}$ be the non-split extension of $\mathrm{PSL}(2, q)$ by $C_{2l}$ in its imprimitive rank 3 representation of degree $2(q+1)$. If $C$ is a binary self-dual*

*code of length* $2(q+1)$ *invariant under* $G$, *then* $C$ *is a* $[2(q+1), q+1, 2]_2$ *code. Moreover,* $C$ *has* $q+1$ *words of weight* 2 *and* $\mathrm{Aut}(C) \cong 2 \wr S_{q+1}$. *The codewords of weight* 2 *are stabilized by a maximal subgroup of* $\mathrm{Aut}(C)$ *of index* $q+1$.

***Proof*** The length and dimension of the codes follow from Table 1 and from the above discussion. Using [3, Proposition 2.15] it can be shown that the minimum weight of $C$ is 2. □

The smallest example of a $\mathrm{PSL}(2, q)$-invariant quasiprimitive imprimitive rank 3 permutation representation occurs when $G = \mathrm{M}_{10} \cong \mathrm{PSL}(2, 9) \cdot 2$ of degree 20 with socle $\mathrm{PSL}(2, 9)$. As alluded to above, there are two inequivalent rank 3 permutation representations of degree 20 in $G$ each with stabilizer $H$ isomorphic to $3{:}S_3 \cdot 2$ and subdegrees 1, 1, and 18. Furthermore, each of these representations produces a unique submodule of dimension 10 which can be proven to be a self-dual code using Result 3. By using [3, Lemma 2.3] one can show that these codes are equivalent. The following example illustrates the situation at hand

***Example 1*** Let $G = \mathrm{M}_{10} \cong \mathrm{PSL}(2, 9) \cdot 2$ be the non-split extension of $\mathrm{PSL}(2, 9)$ by $C_2$. Let $\mathbb{F}_2\Omega$ be the permutation module of dimension 20 associated with the imprimitive rank 3 permutation representation of $G$ of degree 20. Let $C \subset \mathbb{F}_2\Omega$ be a binary self-dual code of length 20 invariant under $G$. Then $C$ is a $[20, 10, 2]_2$ code. Further, $\mathrm{Aut}(C) \cong 2 \wr S_{10}$.

The weight distribution of the code is:

$$A_0 = A_{20} = 1,\ A_2 = A_{18} = 10,\ A_4 = A_{16} = 45,$$
$$A_6 = A_{14} = 120,\ A_8 = A_{12} = 210,\ A_{10} = 252.$$

Since the normalizer $N_G(I(H)) \cong {C_2}^4 \ncong H$, where $I(H)$ denotes the set of involutions of $H$, we conclude by using [3, Proposition 2.15] that the minimum weight of $C$ is 2.

Now, let $G$ be as in Line 3 of Table 1, i.e. $G \cong \mathrm{PSL}(m, q)$. Recall that the field of the group is the finite field of $q$ elements and that $q = p^f$ and $f \geq 1$. By [7, Proposition 4.12 (1)] $G$ is rank 3 if and only if $G$ satisfies the conditions given in part (b) of Remark 2. Moreover, [7, Remark 7] establishes that $G \leq \mathrm{PGL}(m, q) \rtimes \langle \alpha^i \rangle$ with $\alpha$ the Frobenius automorphism and $i$ divides $f$. Here, $d = \min\{j > 0 : \omega^j \in F\}$, with $F \leq GF(q)^*$ and $\omega$ a primitive element of $GF(q)$. Moreover, $d$ divides $gcd(m, q-1)$, $\mathbb{F} = \langle \omega^d \rangle$ and $H = G \cap \mathrm{PGL}(m, q)$, and $[\mathrm{PGL}(m, q){:}H] = d$ with $r = [0, \ldots, d-1]$.

We distinguish two cases depending on the parity of $m$, i.e, $m \geq 3$ and $q$ an odd prime, and $m \geq 4$ even and $q$ an odd prime, respectively.

Here we deal with $m \geq 3$ and odd. The case where $m \geq 4$ is even and $q$ is an odd prime follows immediately afterwards. To this end assume that $s = 2$ with $d = 1$, $r = 0$ and $\lambda = 0$ in Remark 2 (b). Recall from [7, Proposition 4.12 (1)] that under the above assumptions $G = \mathrm{PSL}(m, q)$ produces two inequivalent quasiprimitive imprimitive rank 3 representations of degree $2 \times \frac{q^m-1}{q-1}$. These have subdegrees 1, 1, and $2(\frac{q^m-q}{q-1})$, respectively. Without loss of generality we choose the first representation of that degree, and in Fig. 3 below we give a description of the $\mathbb{F}_2G$-submodule lattice.

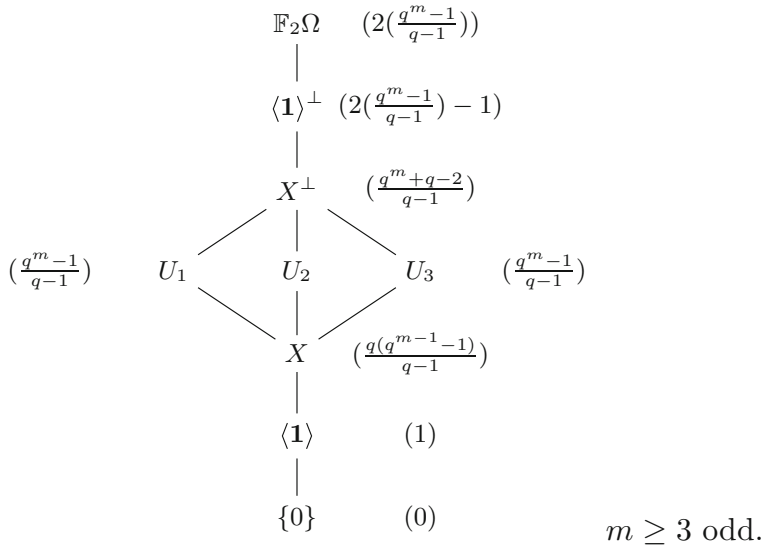


**Fig. 3** Submodule lattice of the quasiprimitive imprimitive rank 3 permutation module of $\mathrm{PSL}(m, q)$ for $m \geq 3$ odd

Now, let $X = C(G, \Omega)$ denote the code of dimension $\frac{q^m-1}{q-1} - 1$ and $X^\perp$ its dual code with dimension $\frac{q^m-1}{q-1} + 1$. Then, it follows from Fig. 3 that between $X$ and $X^\perp$ are the submodules: $X$, $U_1$, $U_2$, $U_3$ and $X^\perp$, with $\dim(U_1) = \dim(U_2) = \dim(U_3) = \frac{q^m-1}{q-1}$.

We note that a similar analysis yields the same results for the other rank 3 representation of this degree. In fact, it can be proven that the binary codes of these two representations are equivalent.

Thus we deduce the following result for the submodules of the $\mathbb{F}_2 G$-module $\mathbb{F}_2\Omega$ of dimension $2 \times \frac{q^m-1}{q-1}$ where $m \geq 3$ is odd and $q$ is an odd prime.

**Proposition 3.5** *The submodules given in Fig. 3 are all $\mathbb{F}_2 G$-submodules of $\mathbb{F}_2\Omega_{2\times\frac{q^m-1}{q-1}}$.*

Applying Result 3 to $U_1$, $U_2$ and $U_3$ we found that only one of these submodules is a self-dual code. Below in Proposition 3.6 we state some of the relevant properties of the unique self-dual code of this representation.

**Proposition 3.6** *Let $G = \mathrm{PSL}(m, q)$ for $m \geq 3$ and $q \geq 3$ both odd, in its imprimitive rank 3 permutation representation of degree $2 \times \frac{q^m-1}{q-1}$. If $C$ is a binary self-dual code of length $2 \times \frac{q^m-1}{q-1}$ invariant under $G$ then $C$ is a $[2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 2]_2$ code. Further, $\mathrm{Aut}(C) \cong 2 \wr S_{\frac{q^m-1}{q-1}}$ and $C$ contains $\frac{q^m-1}{q-1}$ words of weight 2.*

***Proof*** The proof follows virtually the same argument given in the proof of Proposition 3.4. So we omit it. □

***Remark 3*** Notice from Proposition 3.6 that the code $C$ contains $\frac{q^m-1}{q-1}$ words of minimum weight 2. In fact, these codewords span the code.

The smallest example of a $\mathrm{PSL}(m,q)$-invariant quasiprimitive imprimitive rank 3 representation occurs when $(m,q) = (3,3)$, i.e. for $G = \mathrm{PSL}(3,3)$ of degree 26. Recall that in this case we expect to have two equivalent rank 3 representations of degree 26 in $G$. The point stabilizer is a subgroup isomorphic to $3{:}S_3 \cdot 2$ with subdegrees 1, 1, and 24. By the above discussion we obtain a unique self-dual code of dimension 13. Thus we have

***Example 2*** For $G = \mathrm{PSL}(3,3)$ of degree 26 there exists a unique self-dual binary code $C = [26, 13, 2]_2$. Further $\mathrm{Aut}(C) \cong 2 \wr S_{13}$.

The weight distribution of the code is:

$$A_0 = A_{26} = 1,\ A_2 = A_{24} = 13,\ A_4 = A_{22} = 78,\ A_6 = A_{20} = 286,$$
$$A_8 = A_{18} = 715,\ A_{10} = A_{16} = 1287,\ A_{12} = A_{14} = 1716.$$

Notice that there are 13 codewords of minimum weight 2 in $C$. These form a basis for $C$.

Now consider $G = \mathrm{PSL}(m,q)$ and $m \geq 4$ and even. As in the preceding case set $s = 2$, $d = 1$, $r = 0$ and $\lambda = 0$. We distinguish two cases, namely $m = 2u$ and $u$ even, and $m = 2u$ and $u$ odd, respectively.

If $m = 2u$ and $u$ is even, $\mathrm{PSL}(m,q)$ has two isomorphic imprimitive rank 3 actions of degree $2 \times \frac{q^m-1}{q-1}$ when $q = p^2$ and $q \equiv 1 (\mathrm{mod}\, 4)$, and if $m = 2u$ and $u$ is odd the imprimitive rank 3 actions occur for $q \geq 5$ and $q \equiv 1 (\mathrm{mod}\, 4)$. The modulo 2 structure of the permutation module $\mathbb{F}_2\Omega$ as well as its complete $\mathbb{F}_2 G$-submodule lattice are depicted in Fig. 4, where as in the previous cases the dimension of the vector space is in parenthesis.

Note from Fig. 4 that there are three submodules of dimension $\frac{q^m-1}{q-1}$ invariant under $G$. It can be shown by applying Result 3 that all three submodules are self-dual codes. In the result below we state the main obvious properties of these codes. The proof of the result can be obtained by a thorough inspection of the submodule structure of the codes. Observe that for the case when $m = 2u$ and $u$ even, the length is divisible by eight and two of these codes are equivalent doubly-even while the other is a singly-even code.

**Proposition 3.7** *Let $G = \mathrm{PSL}(m,q)$ with $m \geq 4$ even in its imprimitive rank 3 representation of degree $2 \times \frac{q^m-1}{q-1}$.*

(1) *Let $m = 2k$, $k$ an even integer and $q = p^2 \equiv 1 (\mathrm{mod}\, 4)$, $p$ a prime. If $C$ is a binary self-dual code of length $2 \times \frac{q^m-1}{q-1}$ invariant under $G$, then*

   (a) *$C$ is $[2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 2]_2$ code. Moreover, $\mathrm{Aut}(C) \cong 2 \wr S_{\frac{q^m-1}{q-1}}$ and $C$ contains $\frac{q^m-1}{q-1}$ words of weight 2 or*

   (b) *$C$ is a $[2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 4]_2$ doubly-even code.*

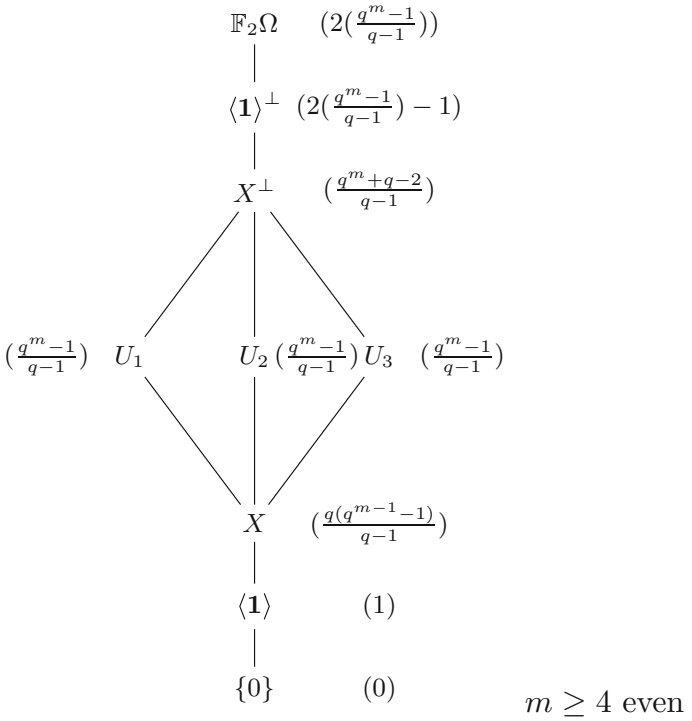


**Fig. 4** Submodule lattice of the quasiprimitive imprimitive rank 3 permutation module of PSL$(m, q)$ for $m \geq 4$ even

(2) *Let* $m = 2k$, $k$ *an odd integer and* $q \equiv 1 \pmod 4$. *If* $C$ *is a binary self-dual code of length* $2 \times \frac{q^m-1}{q-1}$ *invariant under* $G$, *then*

(a) $C$ *is a* $[2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 4]_2$ *code, or*

(b) $C$ *is* $[2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 2]_2$ *code. Moreover,* $\mathrm{Aut}(C) \cong 2 \wr S_{\frac{q^m-1}{q-1}}$ *and* $C$ *contains* $\frac{q^m-1}{q-1}$ *words of weight* 2.

For $m = 2u$, $u$ an even integer and $q^2 \equiv 1 \pmod 4$, the smallest example of a PSL$(m, q)$-invariant quasiprimitive imprimitive rank 3 representation occurs for $G = \mathrm{PSL}(4, 3^2)$ of degree 1640. The action is that of PSL$(4, 9)$ on the cosets of PSL$(3, 9)$:$3^6$ with subdegrees 1, 1, 1638. Applying Result 3 to the permutation module $\mathbb{F}_2\Omega$ of degree 1640 defined by the above imprimitive rank 3 action we obtain 3 submodules of dimension 820, all of which being self-dual as codes. Two of these binary self-dual codes are equivalent and doubly-even and the third is singly-even.

Below in Table 2 we display the submodule lattice of $\mathbb{F}_2\Omega$ where $|\Omega| = 1640$ obtained through computations with MAGMA. Since the table is symmetric about the diagonal we omit the lower half for clarity. In addition, we place a 1 or . in the table

**Table 2** Incidence matrix of the poset of submodules of $\mathbb{F}_2^{1640\times 1}$

| dim | 0 | 1 | 819 | 820 | 820 | 820 | 821 | 1639 | 1640 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | . | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 819 | . | . | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 820 | . | . | . | 1 | . | . | 1 | 1 | 1 |
| 820 | . | . | . | . | 1 | . | 1 | 1 | 1 |
| 820 | . | . | . | . | . | 1 | 1 | 1 | 1 |
| 821 | . | . | . | . | . | . | 1 | 1 | 1 |
| 1639 | . | . | . | . | . | . | . | 1 | 1 |
| 1640 | . | . | . | . | . | . | . | . | 1 |

according to whether or not a submodule is contained in a given module (sometimes itself).

The parameters of the self-dual codes listed in Table 2 are given in

***Example 3*** Let $G = \mathrm{PSL}(4, 9)$ of degree 1640 and let $C$ be a binary self-dual code of length 1640 invariant under $G$. Then $C$ is either a $[1640, 820, 4]_2$ doubly-even code with 335790 words of weight 4 or $C$ is a $[1640, 820, 2]_2$ singly-even code with 820 words of weight 2.

***Proof*** Let $C = [1640, 820, 4]_2$ be a $\mathrm{PSL}(4, 9)$-invariant self-dual code. Via direct enumeration with MAGMA we find that $C$ contains 335790 words of weight 4. Similarly, for $C = [1640, 820, 2]_2$ we obtain 820 words of weight 2. Hence the result. □

For $m = 2u$, $u$ an odd integer and $q \equiv 1 (\mathrm{mod}\, 4)$, the smallest example of a $\mathrm{PSL}(m, q)$-invariant quasiprimitive imprimitive rank 3 representation occurs for $G = \mathrm{PSL}(6, 5)$ of degree 7812. The action is that of $\mathrm{PSL}(6, 5)$ on the cosets of $\mathrm{PSL}(5, 5){:}5^5$ with subdegrees 1, 1, 7810. Applying Result 3 to the permutation module of $\mathrm{PSL}(6, 5)$ over $\mathbb{F}_2$ defined by the above imprimitive rank 3 action constructed by computations with MAGMA we found 3 submodules of dimension 3906, all of which self-dual as codes. Observe that none of these codes is doubly-even since $8 \nmid 7812$.

Below in Table 3 we display the submodule lattice of $\mathbb{F}_2\Omega$ where $|\Omega| = 7812$ obtained through computations with MAGMA.

The parameters of the self-dual codes are given in

***Example 4*** Let $G = \mathrm{PSL}(6, 5)$ of degree 7812. If $C$ denotes a binary self-dual code of length 7812 invariant under $G$ then $C$ is either a $[7812, 3906, 4]_2$ with 7626465 codewords of weight 4 or $C$ is a $[7812, 3906, 2]_2$ singly-even code with 3906 codewords of weight 2.

Now, let $G$ be as in Line 4 of Table 1, ie, $G = \mathrm{PGL}(3, 4)$. There are two inequivalent rank 3 representations of degree 126 in $G$. Using MAGMA we found two self-dual modules of dimension 63 for each representation. Since neither submodule satisfies Result 3 we deduce that there are no self-dual codes of length 126 invariant under $G$.

Let $G = \mathrm{P\Gamma L}(3, 4)$. There are four pairwise equivalent representations of degree 126 in $G$. Two of these representations act rank 12, and are thus excluded. The remaining two are rank 3 on 126 points with subdegrees 1, 5 and 120, respectively. Using

**Table 3** Incidence matrix of the poset of submodules of $\mathbb{F}_2{}^{7812\times 1}$

| dim | 0 | 1 | 3905 | 3906 | 3906 | 3906 | 3907 | 7811 | 7812 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | . | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3905 | . | . | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3906 | . | . | . | 1 | . | . | 1 | 1 | 1 |
| 3906 | . | . | . | . | 1 | . | 1 | 1 | 1 |
| 3906 | . | . | . | . | . | 1 | 1 | 1 | 1 |
| 3907 | . | . | . | . | . | . | 1 | 1 | 1 |
| 7811 | . | . | . | . | . | . | . | 1 | 1 |
| 7812 | . | . | . | . | . | . | . | . | 1 |

MAGMA we constructed the permutation module $\mathbb{F}_2\Omega$ of dimension 126 and obtained two submodules of dimension 63 for each representation. Here too, there are no $G$-invariant self-dual codes of length 126 since neither submodule satisfies Result 3.

Next, consider $G = \mathrm{PSL}(5, 2)$. There are two inequivalent rank 3 representations of degree 248 in $G$. Through computations with MAGMA we found no submodules of dimension 124 in the associated permutation modules.

Let $G = \mathrm{P\Gamma L}(3, 8)$. There are two inequivalent rank 3 representations of degree 2044 in $G$. For each of the two representations we searched for existence of self-dual codes invariant under $G$. We found 16 submodules of dimension 1022, but none of these is a self-dual code.

Finally, let $G = \mathrm{PSL}(3, 2)$ of degree 14. There are two inequivalent imprimitive rank 3 representations of degree 14 in $G$ with stabilizer of a point isomorphic to the alternating group $\mathrm{A}_4$. There are two conjugacy classes of $\mathrm{A}_4$ in $\mathrm{PSL}(3, 2)$ each with subdegrees 1, 1, 12. The reason is because the automorphism group of $\mathrm{PSL}(2, 7) \cong \mathrm{PSL}(3, 2)$ is $\mathrm{PGL}(2, 7)$ and these two conjugacy classes fuse together in $\mathrm{PGL}(2, 7)$.

The associated $\mathbb{F}_2G$ permutation modules give three submodules of dimension 7 each, see Table 4. For the reader's convenience, in Table 4 we give the full $\mathbb{F}_2G$-module structure of the permutation module $\mathbb{F}_2\mathrm{PSL}(3, 2) = \mathbb{F}_2^{14}$ of dimension 14 computed using MAGMA.

We verified with MAGMA that for each permutation module, only one of its 7-dimensional submodule satisfies Result 3. The said submodule is thus a self-dual code. The weight enumerator of the code is

$$A_0 = A_{14} = 1,\ \ A_2 = A_{12} = 7,\ \ A_4 = A_{10} = 21,\ \ A_6 = A_8 = 35.$$

We now have

**Proposition 3.8** *Up to isomorphism there is a unique self-dual code* $C_2(\mathrm{PSL}(3, 2), 14) = [14, 7, 2]_2$ *invariant under* $\mathrm{PSL}(3, 2)$. *Further,* $\mathrm{Aut}(C_2(\mathrm{PSL}(3, 2), 14)) \cong 2 \wr S_7$.

***Proof*** To prove the uniqueness one can appeal to [3, Lemma 2.3], and for the structure of the automorphism we observe that $G$ acts primitively on $7 = \frac{14}{2}$ points, so that

**Table 4** Incidence matrix of the poset of submodules of ${\mathbb{F}_2}^{14\times 1}$

| dim | 0 | 1 | 3 | 4 | 4 | 4 | 5 | 6 | 7 | 7 | 7 | 8 | 9 | 10 | 10 | 10 | 11 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | . | 1 | . | . | 1 | . | 1 | . | . | 1 | . | 1 | . | . | 1 | . | 1 | 1 | 1 |
| 3 | . | . | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | . | . | . | 1 | . | . | 1 | . | 1 | . | . | 1 | . | 1 | . | . | 1 | . | 1 |
| 4 | . | . | . | . | 1 | . | 1 | . | . | 1 | . | 1 | . | . | 1 | . | 1 | 1 | 1 |
| 4 | . | . | . | . | . | 1 | 1 | . | . | . | 1 | 1 | . | . | . | 1 | 1 | . | 1 |
| 5 | . | . | . | . | . | . | 1 | . | . | . | . | 1 | . | . | . | . | 1 | . | 1 |
| 6 | . | . | . | . | . | . | . | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | . | . | . | . | . | . | . | . | 1 | . | . | 1 | . | 1 | . | . | 1 | . | 1 |
| 7 | . | . | . | . | . | . | . | . | . | 1 | . | 1 | . | . | 1 | . | 1 | 1 | 1 |
| 7 | . | . | . | . | . | . | . | . | . | . | 1 | 1 | . | . | . | 1 | 1 | . | 1 |
| 8 | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | 1 | . | 1 |
| 9 | . | . | . | . | . | . | . | . | . | . | . | . | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | 1 | . | 1 |
| 10 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | 1 | 1 | 1 |
| 10 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | 1 | . | 1 |
| 11 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | 1 |
| 13 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | 1 |
| 14 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 |

$\mathrm{Aut}(C_2(G,7)) = S_7$. Hence by using [16, Theorem 3(ii) part (a)] we deduce that $\mathrm{Aut}(C_2(G,14)) \cong 2 \wr S_7 = 2^7{:}S_7$. □

The preceding propositions give the proof of Theorem 1.1 stated in Sect. 1.

***Remark 4*** The general problem of description of the submodule structure for the cross characteristic, and for the defining characteristic of the permutation modules defined by the imprimitive rank 3 groups remains open. In particular, it would be of interest to determine the modulo 2 structure of the rank 3 permutation module of $\mathrm{PSL}(m,q)$ of degree $s \times \frac{q^m-1}{q-1}$ where $s$ is a prime.

## 4 Self-dual doubly-even codes

From the examples of self dual codes constructed in this paper we note that the classification of binary self-dual doubly-even codes invariant under an imprimitive rank 3 permutation group $G$ is reduced to determining those binary self dual codes that satisfy Result 4. In view of this, using results of Sect. 3 we observe that there are possibilities of existence of self-dual doubly-even codes invariant under imprimitive rank 3 permutation groups in the following cases: $\mathrm{Aut}(\mathrm{M}_{12})$ of degree 24 (see Proposition 3.1) and $G = \mathrm{PSL}(m,q)$ of degree $2 \times \frac{q^m-1}{q-1}$ for $m \geq 4$ even and $q = p^2 \equiv 1 (\mathrm{mod}\, 4)$, where $p$ is a prime (see part 1. (b) of Proposition 3.7).

Thus, the classification of binary self-dual doubly-even codes invariant under an imprimitive rank 3 permutation group $G$ is given as follows:

**Proposition 4.1** *Let $C$ be a binary self-dual doubly-even code with $G \leq \mathrm{Aut}(C)$ an imprimitive rank 3 permutation group. Then $C$ is isomorphic to either a code with parameters $[2 \times \frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}, 4]_2$, for $G = \mathrm{PSL}(m, q)$ of degree $2 \times \frac{q^m-1}{q-1}$ and $q = p^2 \equiv 1 (\mathrm{mod}\, 4)$ with $p$ a prime or $C$ is isomorphic to the extended binary Golay code and $G \cong \mathrm{Aut}(\mathrm{M}_{12})$ of degree 24.*

## 5 Self-dual extremal codes

In this section we classify extremal binary self-dual codes admitting finite imprimitive rank 3 groups as permutation automorphism groups.

Due to Mallows-Sloane [19] and Rains [15] a binary self-dual code $C$ of length $n$ and minimum distance $d$ satisfies

$$d \leq \begin{cases} 4\lfloor \frac{n}{24} \rfloor + 4, & \text{if } n \not\equiv 22 (\mathrm{mod}\, 24) \\ 4\lfloor \frac{n}{24} \rfloor + 6, & \text{if } n \equiv 22 (\mathrm{mod}\, 24). \end{cases} \tag{2}$$

A code $C$ is called extremal if equality holds in (2). By Result 4(i) (see also [10]), the length $n$ of a self-dual doubly-even code is a multiple of 8. If $C$ is an extremal doubly-even code then $d \leq 4\lfloor \frac{n}{24} \rfloor + 4$ (see [15]) and $n \leq 3928$, by a result of Zhang [23]. The bound for the length of singly-even extremal self-dual codes is still open. However, the existence of extremal doubly-even codes is known only for small values of $n$; the largest being 136. Thus, there is a large gap between the bound on the length of doubly-even extremal codes and what we can really construct.

A simple check of the minimum distance of the self-dual codes obtained shows that there are no singly-even extremal codes, while there are possibilities of existence of binary self-dual extremal doubly-even codes in the following cases: $\mathrm{Aut}(\mathrm{M}_{12})$ of degree 24, and $\mathrm{PSL}(4, 9)$ of degree 1640. Thus, the classification of extremal codes invariant under an imprimitive rank 3 permutation group $G$ is reduced to determining whether the doubly-even codes of lengths 24, and 1640, respectively are extremal.

For $\mathrm{PSL}(4, 9)$ of degree 1640 we have by Example 3 that $C$ is a $[1640, 820, 4]_2$ code. One can easily verify that $C$ does not satisfy (2) with equality and so it is not extremal. Then it follows by Proposition 3.1 that $C$ is isomorphic to the extended binary Golay code and thus extremal.

With this discussion we have shown that if $C$ is an extremal self-dual doubly-even code and $C$ is invariant under an imprimitive rank 3 permutation group acting transitively on its coordinate positions, then $C$ is isomorphic to the extended binary Golay code. Thus we have

**Theorem 5.1** *Let $C$ be an extremal binary self-dual doubly-even code admitting an imprimitive rank 3 permutation group $G$ as an permutation automorphism group. Then $C$ is isomorphic to the extended binary Golay code and $G$ is isomorphic to* $\mathrm{Aut}(\mathrm{M}_{12})$.

***Remark 5*** In [20] it was proved that there is no extremal binary self-dual code invariant under a primitive rank 3 group of almost simple type. Combining this information with the results obtained in Sect. 3.2 regarding existence (respectively non-existence) of binary self-dual codes invariant under quasiprimitive imprimitive rank 3 groups of almost simple type we deduce the result given a continuation.

**Theorem 5.2** *There is no extremal binary self-dual doubly-even code admitting a quasiprimitive rank* 3 *group G of almost simple type as a permutation automorphism group.*

## 6 Conclusion

In this paper we gave a classification of binary self-dual codes admitting an imprimitive rank 3 permutation group. As a by-product we determined all binary extremal self-dual codes which admit an imprimitive rank 3 group as a permutation group of automorphisms acting transitively on their coordinate positions.

We found that only one known binary extremal self-dual code seems to enjoy the property that a non-solvable group of automorphisms acts rank 3 and transitively on its coordinate positions.

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