On the Capacity of Constrained Permutation Codes for Rank Modulation

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***Abstract*— Motivated by the rank modulation scheme, a recent study by Sala and Dolecek explored the idea of constraint codes for permutations. The constraint studied by them is inherited by the inter-cell interference phenomenon in flash memories, where high-level cells can inadvertently increase the level of low-level cells. A permutation $\sigma \in S_n$ satisfies the single-neighbor k-constraint if $|\sigma(i+1) - \sigma(i)| \leq k$ for all $1 \leq i \leq n-1$. In this paper, this model is extended into two constraints. A permutation $\sigma \in S_n$ satisfies the two-neighbor k-constraint if for all $2 \leq i \leq n-1$, $|\sigma(i) - \sigma(i-1)| \leq k$ or $|\sigma(i+1) - \sigma(i)| \leq k$, and it satisfies the asymmetric two-neighbor k-constraint if for all $2 \leq i \leq n-1$, $\sigma(i-1) - \sigma(i) < k$ or $\sigma(i+1) - \sigma(i) < k$. We show that the capacity of the first constraint is $(1+\epsilon)/2$ in case that $k = \Theta(n^\epsilon)$ and the capacity of the second constraint is 1 regardless for any positive k. We also extend our results and study the capacity of these two constraints combined with error-correcting codes in the Kendall τ-metric.**

***Index Terms*— Error-correcting codes, constrained codes, Kendall τ-metric, permutations, multi-permutations.**

I. Introduction

FLASH memories are, by far, the most important type of non-volatile memory (NVM) in use today. Flash devices are employed widely in mobile, embedded, and mass-storage applications, and the growth in this sector continues at a staggering pace. At the high level, flash memories are comprised of blocks of cells. These cells can have binary values, i.e. they store a single bit, or can have multiple levels and thus can store multiple bits in a cell.

One of the main challenges in flash memories is to exactly program each cell to its designated level. Furthermore, flash memories suffer from the cell leakage problem, by which a charge may leak from the cells and thus cause reading errors [5]. In order to overcome these difficulties, the novel framework of ***rank modulation codes*** was introduced in [11]. Under this setup, the information is represented by permutations which are derived by the relative charge levels of the cells, rather than by their absolute levels. Permutation codes were originally studied five decades ago in the work of Slepian for the transmission of bandlimited signals over Gaussian channels [23] and in the work of Chadwick and Kurz for signal detection over channels with non-Gaussian noise [6].

Another conspicuous property of flash memory, resulting from its rapid growth density, is the appearance of inter-cell interference (ICI). The level of a cell might increase if its neighbor cells are programmed to significantly higher levels [15]. The ICI is caused by the parasitic capacitance between neighboring cells, and in particular, multilevel cell programming is severely influenced by this effect.

Motivated by the rank modulation scheme and the ICI phenomenon, a recent research by Sala and Dolecek [20], [21] proposed the study of constrained codes for permutations. Under this setup, the constraint is invoked over the permutation's symbols. In the model studied in [21], the authors explored the constraint in which the difference between consecutive symbols is upper bounded. In the setting of rank modulation, this constraint prevents the scenario in which a high-level cell affects its low-level neighbor cell. Namely, let S_n be the set of all permutations of length n, then it was said that a permutation $\sigma \in S_n$ satisfies the ***single-neighbor*** k***-constraint*** if $|\sigma_i - \sigma_{i+1}| \leq k$ for all $1 \leq i \leq n-1$. For example, the permutation $\sigma = [3, 1, 2, 4, 5]$ satisfies the single-neighbor 2-constraint but not the single-neighbor 1-constraint. For any positive integers k and n, if $U_{n,k}$ is the set of permutations that meet the single-neighbor k-constraint, then the capacity of this constraint is defined as $C_1(k) = \lim_{n\to\infty} \frac{\log |U_{n,k}|}{\log n!}$. The main result from [21] states that if $k = \Theta(n^\epsilon)$, for some $0 \leq \epsilon \leq 1$, then $C_1(k) = \epsilon$ (in this paper we only use the base 2 logarithm).

In this work, the single-neighbor constraint is naturally extended for two neighbors as it better captures the ICI phenomenon. This extension is applied both symmetrically and asymmetrically. In the symmetric version, as proposed in [21], a permutation satisfies the constraint if the difference between a symbol and one of its neighbors is upper bounded by some prescribed value k. In the asymmetric version, we will constrain the symbols difference only for patterns of the form high-low-high. This constraint is a better modulation for the ICI phenomenon in flash memories since the ICI mainly affects patterns of the form high-low-high and not the other ones [2], [24]. Thus, as in the single-neighbor constraint, we similarly define the capacity of these two constraints and show that if $k = \Theta(n^\epsilon)$, for some $0 \leq \epsilon \leq 1$, then in the symmetric constraint the capacity is $(1+\epsilon)/2$ and in the asymmetric constraint the capacity equals 1 for any positive k.

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The constraints studied in this paper as well as in [21] are effective in reducing the errors caused by the ICI. However, random errors may still happen. While there are several metrics under which error-correcting codes for permutations were studied, we choose to focus on the Kendall τ-metric due to its high applicability to the error behavior in the rank modulation scheme [12] for flash memory. Hence, we will study codes with minimum distance according to the Kendall τ-distance that yet consist of only permutations that satisfy the constraints studied in the paper.

The rest of the paper is organized as follows. In Section II, we introduce the notations and formally define the constraints studied in this paper. Section III studies the capacity of the symmetric constraint and Section IV studies the capacity of the asymmetric constraint. In Section V, we extend our results and study the capacity of these two constraints combined with error-correction codes in the Kendall τ-metric. We conclude our results in Section VI.

II. Definitions and Notations

In this section we formally define the constraints studied in the paper and introduce some of the notations and tools that we will use to compute their capacity.

A permutation on a finite set X is a bijection $\sigma : X \to X$. Denote by $[n]$ the set of n integers $\{1, 2, \ldots, n\}$. For two integers a, b, where $a < b$, denote by $[a, b]$ the set of $b - a + 1$ integers $\{a, a+1, a+2, \ldots, b\}$. Let S_n be the set of all permutations on $[n]$ and let $S([a, b])$ be the set of all permutations on $[a, b]$. We denote a permutation $\sigma \in S_n$ ($\sigma \in S([a, b])$, respectively) by $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(n)]$ ($\sigma = [\sigma(a), \sigma(a+1), \ldots, \sigma(b)]$, respectively). The inverse of $\sigma \in S_n$ is denoted by $\sigma^{-1} = [\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(n)]$, where $\sigma^{-1}(i) = j$ if $\sigma(j) = i$.

We use permutations of length n in order to represent the ranking of n flash memory cells in the rank modulation scheme. In particular, the n flash memory cells are represented by a permutation $\sigma \in S_n$, where $\sigma(i)$ corresponds to the ranking of the i-th cell in an increasing order. For example, if $\sigma(i) = 1$ then the i-th cell has the lowest charge level while $\sigma(i) = n$ implies that the i-th cell stores the highest charge level. Under this definition of ranking, if $\sigma(i+1) - \sigma(i)$ is high then the charge level of the $(i+1)$-st cell is much higher than the charge level of the i-th cell.

Remark 1: Note that there are two alternatives to represent the cells' rankings by a permutation. The first one is the method we use in this paper where $\sigma(i)$ corresponds to the ranking of the i-th cell. In the second approach $\sigma(i)$ is the index of the cell with the i-th rank. Hence, the permutation that represents some n cells according to the second approach is simply the inverse of the permutation that represents these cells according to the first approach. While the two representations are dual to each other, we chose the first one for the convenience of describing the constraints in our work.

Definition 2: Let n and k be positive integers. A permutation $\sigma \in S_n$ is said to satisfy the ***two-neighbor k-constraint*** if for all i, $2 \leq i \leq n-1$, $|\sigma(i) - \sigma(i-1)| \leq k$ or $|\sigma(i+1) - \sigma(i)| \leq k$. Denote by $A_{n,k}$ the set of all permutations in S_n that satisfy the two-neighbor k-constraint. A ***two-neighbor k-constrained code*** is a subset of $A_{n,k}$. Finally, for every function $k : \mathbb{N} \to \mathbb{N}$, the ***capacity*** of the two-neighbor k-constraint is defined as

$$C(k) = \limsup_{n\to\infty} \frac{\log |A_{n,k(n)}|}{\log n!}. \quad (1)$$

Example 3: If the charge levels of seven flash memory cells are given by $(0.2, 1, 2.5, 0.75, 1.5, 1.9, 0.5)$ then the representation of these cells is given by the permutation $\sigma = [1, 4, 7, 3, 5, 6, 2]$. The permutation σ satisfies the two-neighbor 3-constraint but not the two-neighbor 2-constraint.

If $k \geq n-1$ then since the absolute value of the difference between two distinct elements of $[n]$ is at most $n-1$, it follows that $A_{n,k} = S_n$. In fact, $A_{n,k} = S_n$ even for $k = n-2$. This property holds since the only two distinct elements that admit a difference which is higher than k are 1 and n, and therefore for every $2 \leq i \leq n-1$, $|\sigma(i) - \sigma(i-1)| \leq k$ or $|\sigma(i+1) - \sigma(i)| \leq k$.

The purpose of the two-neighbor constraint is to combat the ICI problem by eliminating high differences between the rankings of a cell with both of its neighbors. However, this constraint does not distinguish between high-low-high and low-high-low patterns and thus eliminates them both. A weaker constraint which may fit better to the ICI phenomena is defined next.

Definition 4: Let n and k be positive integers. A permutation $\sigma \in S_n$ is said to satisfy the ***asymmetric two-neighbor k-constraint*** if for all i, $2 \leq i \leq n-1$, $\sigma(i-1) - \sigma(i) \leq k$ or $\sigma(i+1) - \sigma(i) \leq k$. The set of all permutations that satisfy the asymmetric two-neighbor k-constraint is denoted by $B_{n,k}$. An ***asymmetric two-neighbor k-constrained code*** is a subset of $B_{n,k}$ and for every function $k : \mathbb{N} \to \mathbb{N}$, the constraint's capacity is defined as

$$\widetilde{C}(k) = \limsup_{n\to\infty} \frac{\log |B_{n,k(n)}|}{\log n!}.$$

Example 5: The permutation $\sigma = [1, 4, 7, 3, 5, 6, 2]$ from Example 3 satisfies the asymmetric two-neighbor 2-constraint but not the asymmetric two-neighbor 1-constraint.

Remark 6: We restrict our discussion on the capacities of both constraints only for functions $k : \mathbb{N} \to \mathbb{N}$ such that $k = \Theta(n^{\epsilon})$, for some $0 \leq \epsilon \leq 1$. Also, we use in the capacity definitions the supremum limit versions since the limits do not necessarily exist. However, we shall later see that if $k = \Theta(n^{\epsilon})$ then these limits exist.

Note, that every permutation which satisfies the two-neighbor k-constraint satisfies the asymmetric two-neighbor k-constraint as well, thus $C(k) \leq \widetilde{C}(k)$.

In the construction of two-neighbor k-constrained codes we will use multi-permutations, which are natural generalization of permutations. A ***balanced multi-set*** $\mathcal{M}_{\ell,m} = \{1^m, 2^m, \ldots, \ell^m\}$ is a collection of the elements in $[\ell]$, each appears m times. A multi-permutation on $\mathcal{M}_{\ell,m}$ is a function $\sigma : [\ell m] \to [\ell]$ such that $|\{j : \sigma(j) = i\}| = m$, for all $i \in [\ell]$. As for permutations, σ will be represented by $[\sigma(1), \sigma(2), \ldots, \sigma(\ell m)]$. The set of all multi-permutations over $\mathcal{M}_{\ell,m}$ is denoted by $P_{\ell,m}$. This definition can be extended for multi-sets which are not balanced, however we will not need this generalization for our purposes. For a multi-permutation $\sigma \in P_{\ell,m}$, we distinguish between appearances of the same number in σ, by their positions in σ. We consider the increasing order of these positions. For every $i \in [\ell]$ and $r \in [m]$, we denote by i_r the rth appearance of i in σ. By abuse of notation, we sometimes write for $i \in [\ell], r \in [m], \sigma(j) = i_r$ and $j = \sigma^{-1}(i_r)$ to indicate that the r-th appearance of i is in the j-th position of σ.

Example 7: For the multi-set $\mathcal{M}_{3,2} = \{1^2, 2^2, 3^2\}$, we have that $\sigma = [1, 3, 1, 2, 3, 2]$ is a multi-permutation in $P_{3,2}$, and for example $\sigma(3) = 1_2$, and $3 = \sigma^{-1}(1_2)$.

Note, that multi-permutations, besides of being a tool in our solutions, find interest also in flash memory applications. The rank modulation scheme was recently generalized such that multiple cells can hold the same rank and thus represent a multi-permutation; see e.g. [9], [10]. As a consequence, error-correction codes for multi-permutations have attracted attention as well [4], [22]. Hence, the generalization of the aforementioned constraints and similar ones for multi-permutations is also very important and interesting, however is out of the scope of this paper.

III. The Two-Neighbor Constraint

In this section we study the two-neighbor constraint and in particular find its capacity. This will be done first by a construction of two-neighbor k-constrained codes which provides a lower bound on the capacity. The construction is based upon assigning permutations into a special family of multi-permutations. Then, we will present an upper bound on the size of the set $A_{n,k}$ which will result with an upper bound on the capacity that coincides with the lower bound by the construction.

For a multi-permutation $\rho \in P_{\ell,m}$ and permutations $\gamma_1, \gamma_2, \dots, \gamma_\ell$, such that $\gamma_i \in S([(i-1)m+1, im])$ for $i \in [\ell]$, the ***assignment*** of the permutations $\gamma_1, \gamma_2, \dots, \gamma_\ell$ in the multi-permutation ρ is the permutation $\sigma = \rho(\gamma_1, \gamma_2, \dots, \gamma_\ell) \in S_{\ell m}$ defined as follows. For all $1 \le j \le n$, if $\rho(j) = i_r$ for some $i \in [\ell]$ and $r \in [m]$ then $\sigma(j) = \gamma_i((i-1)m + r)$. In other words, σ is obtained from ρ by replacing the r-th appearance of i with the r-th element of γ_i.

Example 8: If $\rho = [1, 2, 1, 3, 2, 3] \in P_{3,2}$, $\gamma_1 = [2, 1]$, $\gamma_2 = [3, 4]$, and $\gamma_3 = [6, 5]$ then $\rho(\gamma_1, \gamma_2, \gamma_3) = [2, 3, 1, 6, 4, 5]$.

The following lemma will be useful for the construction of the two-neighbor k-constrained codes presented in this section.

Lemma 9: Let $\rho_1, \rho_2 \in P_{\ell,m}$, $\gamma_1, \gamma_2, \dots, \gamma_\ell$, and $\delta_1, \delta_2, \dots, \delta_\ell$, where $\gamma_i, \delta_i \in S([(i-1)m + 1, im])$, for all $i \in [\ell]$. For $\sigma = \rho_1(\gamma_1, \gamma_2, \dots, \gamma_\ell)$ we have that $\sigma \in S_{\ell m}$. Moreover, $\sigma = \rho_2(\delta_1, \delta_2, \dots, \delta_\ell)$ if and only if $\rho_1 = \rho_2$ and $\gamma_i = \delta_i$, for all $i \in [\ell]$.

Proof: Clearly, the assignment of the permutations $\gamma_1, \gamma_2, \dots, \gamma_\ell$ in the multi-permutation ρ_1 results in a multi-permutation of length ℓm. Since for every $i \in [\ell]$, γ_i is a permutation on $[(i-1)m+1, im]$, it follows that every element in $[\ell m]$ appears exactly once in σ and thus $\sigma \in S_{\ell m}$.

If $\rho_1 = \rho_2$ and $\gamma_i = \delta_i$, for all $i \in [\ell]$ then $\sigma = \rho_2(\delta_1, \delta_2, \dots, \delta_\ell)$. For the other direction, assume that $\sigma = \rho_2(\delta_1, \delta_2, \dots, \delta_\ell)$. Let $\sigma(j) = s$ for some $j \in [\ell m]$ and let i be the unique element in $[\ell]$ for which $(i-1)m + 1 \le s \le im$. By the definition of σ we have that $\rho_1(j) = \rho_2(j) = i$. Hence, $\rho_1(j) = \rho_2(j)$ for all $j \in [n]$, and therefore $\rho_1 = \rho_2$.

Let $r \in [m]$ such that $\gamma_i((i-1)m + r) = s$. Then $\sigma(j) = \gamma_i((i-1)m + r)$, and by the definition of $\rho_1(\gamma_1, \gamma_2, \dots, \gamma_\ell)$ it follows that $\rho_1(j) = i_r$. Since $\rho_1 = \rho_2$, it follows that $\rho_2(j) = i_r$ and by the definition of $\rho_2(\delta_1, \delta_2, \dots, \delta_\ell)$ we have that $\sigma(j) = \delta_i((i-1)m + r) = s$. Hence, $\gamma_i((i-1)m + r) = \delta_i((i-1)m + r)$, for all $i \in [\ell]$ and for all $r \in [m]$, and therefore $\gamma_i = \delta_i$, for all $i \in [\ell]$. ∎

For an even integer m, the set $D_{\ell,m} \subseteq P_{\ell,m}$ is defined as follows. A multi-permutation $\rho \in P_{\ell,m}$ belongs to $D_{\ell,m}$ if $\rho(2j-1) = \rho(2j)$ for every $j \in [\ell m/2]$. Note, that $D_{\ell,m}$ has the same size as $P_{\ell,m/2}$, i.e. $|D_{\ell,m}| = |P_{\ell,m/2}| = \frac{(\ell m/2)!}{(m/2)!^\ell}$.

Example 10: The multi-permutation $\rho = [1, 1, 2, 2, 2, 2, 3, 3, 1, 1, 3, 3]$ belongs to $D_{3,4}$ since $\rho(1) = \rho(2)$, $\rho(3) = \rho(4)$, and so on.

Next, we present a construction of two-neighbor k-constrained codes.

Construction 11: Let $n = \ell(k+1)$, where k is an odd positive integer and ℓ is a positive integer. Let $\mathcal{C}_{n,k}^{sym} \subseteq S_n$ be the code consisting of all the permutations $\sigma \in S_n$ of the form $\sigma = \rho(\gamma_1, \gamma_2, \dots, \gamma_\ell)$, where $\rho \in D_{\ell,k+1}$ and $\gamma_i \in S([(i-1)(k+1)+1, i(k+1)])$, for all $i \in [\ell]$. That is,

$$\mathcal{C}_{n,k}^{sym} = \left\{ \rho(\gamma_1, \dots, \gamma_\ell) : \begin{array}{c} \rho \in D_{\ell,k+1}, \forall i \in [\ell] \\ \gamma_i \in S([(i-1)(k+1)+1, i(k+1)]) \end{array} \right\}.$$

The correctness of Construction 11 as well as the code cardinality are stated in the next lemma.

Lemma 12: Let n, k, ℓ be as specified in Construction 11. Then, the code $\mathcal{C}_{n,k}^{sym}$ is a two-neighbor k-constrained code and its cardinality is

$$|\mathcal{C}_{n,k}^{sym}| = \frac{\left(\frac{n}{2}\right)!(k+1)!^\ell}{\left(\frac{k+1}{2}\right)!^\ell}.$$

Proof: Let $\sigma \in \mathcal{C}_{n,k}^{sym}$. There exist $\rho \in D_{\ell,k+1}$ and $\gamma_1, \gamma_2, \dots, \gamma_\ell$, where $\gamma_i \in S([(i-1)(k+1)+1, i(k+1)])$ for all $i \in [\ell]$, such that $\sigma = \rho(\gamma_1, \gamma_2, \dots, \gamma_\ell)$. Let $2 < j \le n-1$ be an odd integer and assume that $\rho(j) = i_r$ for some $i \in [\ell]$ and $r \in [k+1]$. By the definition of $D_{\ell,k+1}$, it follows that $\rho(j+1) = i_{r+1}$ and in particular, $r \le k$. Hence, $\sigma(j) = \gamma_i((i-1)(k+1) + r) \in [(i-1)(k+1)+1, i(k+1)]$ and similarly $\sigma(j+1) = \gamma_i((i-1)(k+1) + r + 1) \in [(i-1)(k+1)+1, i(k+1)]$. It follows that $|\sigma(j) - \sigma(j+1)| \le k$. The case where j is even is handled similarly with respect to the symbol in position $j-1$. Thus, σ satisfies the two-neighbor k-constraint.

For the computation of the cardinality of $\mathcal{C}_{n,k}^{sym}$, note that by Lemma 9 it follows that every choice of $\rho \in D_{\ell,k+1}$ and $\gamma_1, \gamma_2, \dots, \gamma_\ell$, where $\gamma_i \in S([(i-1)(k+1)+1, i(k+1)])$ for all $i \in [\ell]$, generates a different codeword of the form $\rho(\gamma_1, \gamma_2, \dots, \gamma_\ell)$. Therefore,

$$|\mathcal{C}_{n,k}^{sym}| = |D_{\ell,k+1}| \cdot (k+1)!^\ell = \frac{\left(\frac{n}{2}\right)!(k+1)!^\ell}{\left(\frac{k+1}{2}\right)!^\ell}.$$

∎

Note, that the structure of the construction also applies the existence of efficient encoding and decoding mappings for this code. This follows from the observation that we need to encode and decode over a set of multi-permutations ($D_{\ell,k+1}$ which is equivalent to the set of multi-permutations $P_{\ell,(k+1)/2}$) and over the set of permutations $S([(i-1)(k+1)+1, i(k+1)])$, for all $i \in [\ell]$. Encoding and decoding for multi-permutations and permutations can be conducted efficiently, for example by using enumerative encoding methods [7], [18].

Even though Construction 11 provides two-neighbor k-constrained codes only for the case where k is odd, it can be easily modified for the case that k is even as well. In any event, we will not need this modification in order to calculate a lower bound on the capacity, which is stated in the next theorem.

Theorem 13: If $k = \Theta(n^\epsilon)$ for some $0 \le \epsilon \le 1$, then $C(k) \ge \frac{1+\epsilon}{2}$.

Proof: If k is an odd integer and if n is divisible by $k+1$ then by Lemma 12 we have

$$|A_{n,k}| \ge \frac{\left(\frac{n}{2}\right)!(k+1)!^{\frac{n}{k+1}}}{\left(\frac{k+1}{2}\right)!^{\frac{n}{k+1}}}$$

and by the bounds $(n/e)^n \le n! \le n^n$, see e.g. [26, p. 54], we have that

$$|A_{n,k}| \ge \frac{\left(\frac{n}{2e}\right)^{\frac{n}{2}}\left(\frac{k+1}{e}\right)^n}{\left(\frac{k+1}{2}\right)^{\frac{n}{2}}} \ge c^n(n(k+1))^{\frac{n}{2}}, \qquad (2)$$

where c is some constant.

Let $k = \Theta(n^\epsilon)$ and assume w.l.o.g. that $k(n)$ is odd for all $n \in \mathbb{N}$ (otherwise, we can define $\widetilde{k}(n) = k(n)$ if $k(n)$ is odd and $\widetilde{k}(n) = k(n) - 1$ if $k(n)$ is even and prove the lemma for $\widetilde{k}$). We continue the proof by distinguishing between two cases.

Case 1 ($0 \le \epsilon < 1$): For every n let $\nu_n \in [n]$ be the largest integer which is divisible by $(k+1)$, that is $\nu_n = (k+1)\lfloor n/(k+1)\rfloor$. Then $\nu_n \ge \max\{n-k, k\}$, and since $\epsilon < 1$, it follows that $\nu_n = n - o(n)$ and $k = \Theta({\nu_n}^\epsilon)$. By (2) it follows that

$$|A_{\nu_n,k}| \ge c_1^{\nu_n}\nu_n^{\left(\frac{1+\epsilon}{2}\right)\nu_n}, \qquad (3)$$

where c_1 is a constant. Then

$$C(k) = \limsup_{n\to\infty}\frac{\log|A_{n,k}|}{\log n!} \ge \limsup_{n\to\infty}\frac{\log|A_{\nu_n,k}|}{\log \nu_n!}\frac{\log \nu_n!}{\log n!}, \qquad (4)$$

where the inequality follows from $|A_{\nu_n,k}| \le |A_{n,k}|$. Since $\nu_n = n - o(n)$ it follows that

$$\lim_{n\to\infty}\frac{\log \nu_n!}{\log n!} = 1. \qquad (5)$$

By combining (3), (4), and (5) we have

$$C(k) \ge \lim_{n\to\infty}\frac{\log\left(c_1^{\nu_n}\nu_n^{\left(\frac{1+\epsilon}{2}\right)\nu_n}\right)}{\log \nu_n!} = \frac{1+\epsilon}{2}.$$

Case 2: $\epsilon = 1$. In this case $k(n) \ge \frac{n}{q}$ for some constant integer q and for sufficiently large n, and we let $\mu_n = qn$. Then $k(\mu_n) \ge n$ and

$$C(k) \ge \limsup_{n\to\infty}\frac{\log|A_{n,k(n)}|}{\log n!} \ge^{(a)} \limsup_{n\to\infty}\frac{\log|A_{\mu_n,k(\mu_n)}|}{\log \mu_n!}$$
$$\ge^{(b)} \limsup_{n\to\infty}\frac{\log|A_{qn,n-1}|}{\log(qn)!} \ge^{(c)} \lim_{n\to\infty}\frac{\log(c'^n n^{qn})}{\log(qn)!} = 1,$$

for some constant c'. Note that inequality (a) follows from the fact that $\log|A_{\mu_n,k(\mu_n)}|/\log\mu_n!$ is a subsequence of $\log|A_{n,k}|/\log n!$. Inequality (b) follows from $k(\mu_n) \ge n > n-1$ and the fact that $|A_{n,k}|$ is a monotone increasing function of k. Inequality (c) follows from applying (2) on the size of $A_{qn,n-1}$.

Thus, we showed that $C(\epsilon) \ge \frac{1+\epsilon}{2}$ for all $k = \Theta(n^\epsilon)$, $0 \le \epsilon \le 1$. ∎

In order to derive an upper bound on the capacity $C(k)$ we show an upper bound on the size of $A_{n,k}$.

Lemma 14: For all positive integers n, k such that $k < n$,

$$|A_{n,k}| \le 4^{n-1}k^{\frac{n}{2}}n^{\frac{n}{2}+1}.$$

Proof: Let $\psi : A_{n,k} \to \mathbb{Z}^n$ be the following mapping. For a permutation $\sigma \in A_{n,k}$, $\psi(\sigma) = \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$, where $x_1 = \sigma(1)$ and for each i, $2 \le i \le n$, $x_i = \sigma(i) - \sigma(i-1)$. Clearly, ψ is an injection and therefore, the size of the set $A_{n,k}$ is equal to the size of the image of ψ, $\psi(A_{n,k}) = \{\psi(\sigma) : \sigma \in A_{n,k}\}$. We will show an upper bound on the size of $\psi(A_{n,k})$.

Let $\mathbf{x} = \psi(\sigma)$ for some $\sigma \in A_{n,k}$. For any position j, $2 \le j \le n-1$, either $|\sigma(j)-\sigma(j-1)| \le k$ or $|\sigma(j+1)-\sigma(j)| \le k$. Therefore, at least $\left\lfloor\frac{n-1}{2}\right\rfloor$ of the $n-1$ elements $x_2, x_3, \ldots, x_n$ are in the range $[-k, k] \setminus \{0\}$. Let $I \subseteq [2, n]$ be a set with at least $\left\lfloor\frac{n-1}{2}\right\rfloor$ elements and let D_I be the set of all vectors $\mathbf{x} \in \psi(A_{n,k})$ for which $x_i \in [-k, k] \setminus \{0\}$, for every $i \in I$ and $x_j \in [-n, n] \setminus [-k, k]$, for every $j \in [2, n] \setminus I$. Then,

$$|\psi(A_{n,k})| \le \sum_{I\subseteq[2,n]:\ |I| \ge \left\lfloor\frac{n-1}{2}\right\rfloor} |D_I|. \qquad (6)$$

For each $i \in I$ there are $2k$ choices for x_i and for each $j \in [2, n] \setminus I$ there are at most $2(n-k) < 2n$ choices for x_j. Finally, there are n choices for x_1. Therefore,

$$|D_I| \le n \cdot (2k)^{\left\lfloor\frac{n-1}{2}\right\rfloor} \cdot (2n)^{\left\lceil\frac{n-1}{2}\right\rceil} = 2^{n-1}k^{\left\lfloor\frac{n-1}{2}\right\rfloor}n^{\left\lceil\frac{n-1}{2}\right\rceil+1}.$$

Since the number of ways to choose the set I is at most 2^{n-1}, according to (6), the following upper bound on the cardinality of $A_{n,k}$ and $\psi(A_{n,k})$ is derived

$$|A_{n,k}| = |\psi(A_{n,k})| \le 2^{n-1} \cdot 2^{n-1}k^{\left\lfloor\frac{n-1}{2}\right\rfloor}n^{\left\lceil\frac{n-1}{2}\right\rceil+1}$$
$$\le 4^{n-1}k^{\frac{n}{2}}n^{\frac{n}{2}+1}.$$

∎

As a result of the last lemma we derive the following theorem which provides an upper bound on the capacity.

Theorem 15: If $k = \Theta(n^\epsilon)$ for some $0 \le \epsilon \le 1$, then $C(k) \le \frac{1+\epsilon}{2}$.

Proof: By Lemma 14 it follows that there exists some constant c such that $|A_{n,k}| \le c^n n^{\frac{(1+\epsilon)n}{2}+1}$ for every sufficiently large n, and hence

$$C(k) = \limsup_{n\to\infty} \frac{\log |A_{n,k}|}{\log n!} \le \lim_{n\to\infty} \frac{\log(c^n n^{\frac{(1+\epsilon)n}{2}+1})}{\log n!} = \frac{1+\epsilon}{2}.$$

■

The following corollary, which is an immediate result of Theorems 13 and 15, summarizes the discussion of this section.

Corollary 16: If $k = \Theta(n^{\epsilon})$ for some $0 \le \epsilon \le 1$, then $C(k) = \frac{1+\epsilon}{2}$.

IV. The Asymmetric Two-Neighbor Constraint

In this section we find the capacity of the asymmetric two-neighbor constraint. Our main result states that $\widetilde{C}(k) = 1$ for every function $k : \mathbb{N} \to \mathbb{N}$. Since the capacity is at most 1, and the capacity is nondecreasing when k increases, we will need to show that $\widetilde{C}(1) = 1$ (here, 1 corresponds to the constant function $k(n) = 1$ for all $n \in \mathbb{N}$). This will be done by a construction of an asymmetric two-neighbor 1-constrained code that confirms this capacity result.

A position i, $2 \le i \le n-1$, is called a ***valley*** in a permutation $\sigma \in S_n$ if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) < \sigma(i+1)$. For example, in the permutation $\sigma = [4, 7, 5, 6, 1, 2, 3]$, the third and fifth positions are valleys. Note, that a permutation satisfies the asymmetric two-neighbor 1-constraint if and only if for every valley i of σ either $\sigma(i+1) = \sigma(i)+1$ or $\sigma(i-1) = \sigma(i)+1$. Hence, in order to generate codewords that satisfy the asymmetric two-neighbor 1-constraint, we first partition the high value numbers to groups, which we order in runs of increasing elements followed by runs of decreasing elements. We then use the low value numbers to separate the groups by pairs of consecutive elements. In this way for a valley i, $\sigma(i)$ must be a low value number and $\sigma(i+1) = \sigma(i)+1$ or $\sigma(i-1) = \sigma(i)+1$.

For a nonempty set of integers I, let $I^{\nearrow}, I^{\searrow}$, denote the ordering of all elements in I according to their increasing, decreasing order, respectively. For the construction of an asymmetric two-neighbor 1-constrained code we will need the code $\mathcal{C}_{r,1}^{sym}$, where r is even, from Construction 11. Recall that a permutation $\pi \in \mathcal{C}_{r,1}^{sym}$ is of the form

$$\pi = \rho(\gamma_1, \gamma_2, \dots, \gamma_{\frac{r}{2}}),$$

where $\rho \in P_{r,2}$ such that $\rho(2i-1) = \rho(2i)$ and $\gamma_i \in S([2i-1, 2i])$, for all $1 \le i \le \frac{r}{2}$. In other words, for every j, $1 \le j \le \frac{r}{2}$, there exists $1 \le i \le \frac{r}{2}$ such that $\{\pi(2j-1), \pi(2j)\} = \{2i-1, 2i\}$.

Construction 17: Let n be an even integer and let r be an integer, $3 \le r \le \frac{n}{2}$. If r is even, let the code $\mathcal{C}_r \subset S_n$ be defined as follows. A permutation $\sigma \in S_n$ belongs to $\mathcal{C}_r$ if there exists a partition of the set $[r-1, n]$ into r nonempty sets $I_1, I_2, \dots, I_r$, and a permutation $\pi \in \mathcal{C}_{r-2,1}^{sym}$ such that

$$\sigma = [I_1^{\nearrow}, I_2^{\searrow}, \pi(1), \pi(2), I_3^{\nearrow}, I_4^{\searrow}, \dots, \pi(r-3), \pi(r-2), I_{r-1}^{\nearrow}, I_r^{\searrow}].$$

For an odd r, let the code $\mathcal{C}_r \subset S_n$ be defined in a similar way. A permutation $\sigma \in S_n$ belongs to $\mathcal{C}_r$ if there exists a partition of the set $[r, n]$ into r nonempty sets $I_1, I_2, \dots, I_r$, and a permutation $\pi \in \mathcal{C}_{r-1,1}^{sym}$ such that

$$\sigma = [I_1^{\nearrow}, I_2^{\searrow}, \pi(1), \pi(2), I_3^{\nearrow}, I_4^{\searrow}, \dots, \pi(r-2), \pi(r-1), I_r^{\nearrow}].$$

Finally, let $\mathcal{C}_n^{asym} \subset S_n$ be the code

$$\mathcal{C}_n^{asym} = \bigcup_{r=3}^{n/2} \mathcal{C}_r.$$

Example 18: For $n = 14$ and $r = 5$, let $I_1 = \{5, 8, 10\}$, $I_2 = \{6, 12\}$, $I_3 = \{7, 15\}$, $I_4 = \{9, 13\}$, $I_5 = \{11, 14\}$ be a partition of $[5, 14]$ into five nonempty sets and let $\pi = [4, 3, 1, 2]$. Note, that $\pi = \rho(\gamma_1, \gamma_2)$ where $\rho = [2, 2, 1, 1]$, $\gamma_1 = [1, 2] \in S([1, 2])$, and $\gamma_2 = [4, 3] \in S([3, 4])$, hence, π is a codeword in $\mathcal{C}_{4,1}^{sym}$. The permutation $\sigma \in \mathcal{C}_5$ of the form $\sigma = [I_1^{\nearrow}, I_2^{\searrow}, \pi(1), \pi(2), I_3^{\nearrow}, I_4^{\searrow}, \pi(3), \pi(4), I_5^{\nearrow}]$ is $\sigma = [5, 8, 10, 12, 6, 4, 3, 7, 15, 13, 9, 1, 2, 11, 14]$. Note, that σ can also be obtained from other partitions such as $\widetilde{I}_1 = \{5, 8, 10, 12\}$, $\widetilde{I}_2 = \{6\}$, and $\widetilde{I}_i = I_i$, for all $3 \le i \le 5$.

The next lemma will be used in proving the correctness of Construction 17.

Lemma 19: Let m be an integer, $1 \le m \le \frac{n-4}{4}$. Then, every permutation $\sigma \in \mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$ has exactly m valleys.

Proof: Let $\sigma \in \mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$. Then σ is formed as described in Construction 17 by a permutation $\pi \in \mathcal{C}_{2m,1}^{sym}$ and a partition of the set $[2m+1, n]$, $I_1, I_2, \dots, I_r$, where $r \in \{2m+1, 2m+2\}$. If $\sigma(i) \in I_s$ for some $2 \le i \le n-1$ and $1 \le s \le r$, then either $\sigma(i-1) < \sigma(i)$ or $\sigma(i+1) < \sigma(i)$, and hence i cannot be a valley in σ. Therefore, if i is a valley then $\sigma(i) = \pi(j)$ for some $1 \le j \le 2m$. Since for every υ, $1 \le \upsilon \le \frac{n}{2}$, there exists an u, $1 \le u \le \frac{n}{2}$, such that $\{\pi(2\upsilon-1), \pi(2\upsilon)\} = \{2u-1, 2u\}$ and since $\pi(2\upsilon-1)$ and $\pi(2\upsilon)$ are adjacent elements in σ, it follows that i is a valley in σ if and only if $\pi(j)$ is odd. Hence, every element in $\mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$ has exactly m valleys. ■

The correctness of the construction of the code $\mathcal{C}_n^{asym}$ is proved in the next lemma.

Lemma 20: For all $n \ge 1$, the code $\mathcal{C}_n^{asym}$ is an asymmetric two-neighbor 1-constrained code.

Proof: Let $\sigma \in \mathcal{C}_n^{asym}$ and let m be the number of valleys in σ. By Lemma 19 it follows that $\sigma \in \mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$. Also, by the proof of Lemma 19 and according to Construction 17 it follows that there exists a permutation $\pi \in \mathcal{C}_{2m,1}^{sym}$ such that the valleys of σ are in positions i, where $\sigma(i) = \pi(j)$ for some $1 \le j \le 2m$ such that $\pi(j)$ is odd. It follows that either $\sigma(i-1) = \sigma(i)+1$ or $\sigma(i+1) = \sigma(i)+1$. Then the valleys in σ do not violate the asymmetric two-neighbor 1-constraint and therefore σ satisfies the asymmetric two-neighbor 1-constraint. ■

Next, we will analyze a lower bound on the cardinalities of the codes from Construction 17. First, we use the following observation.

Lemma 21: Let n be an integer such that $n \equiv 0 \pmod 4$, let $\sigma \in \mathcal{C}_n^{asym}$, and let m be the number of valleys in σ. Then there exist at most 2^{m+1} different ways to obtain σ as described in Construction 17.

Proof: By Lemma 19 it follows that σ belongs to $\mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$. Let $i_1 < i_2 < \cdots < i_{2m}$ be the $2m$ positions in which the elements of the set $[2m]$ appear in σ. If $\pi \in \mathcal{C}^{sym}_{2m,1}$ is a permutation from which σ is obtained as described in Construction 17 then $\pi = [\sigma(i_1), \sigma(i_2), \dots, \sigma(i_{2m})]$, and hence π is uniquely determined by σ. If $I_1, I_2, \dots, I_{2m+1}, I_{2m+2}$ is a partition of the set $[2m+1, n]$ into either $2m+1$ or $2m+2$ nonempty sets (we allow only the set I_{2m+2} to be empty), then $[I_1^{\nearrow}, I_2^{\searrow}] = [\sigma(1), \sigma(2), \dots, \sigma(i_1 - 1)]$. Let j, $1 \le j \le i_1 - 1$ be the position such that $\sigma(j) \ge \sigma(i)$ for all $1 \le i \le i_1 - 1$. If $\sigma(j) \in I_1$ then $I_1 = \{\sigma(1), \sigma(2), \dots, \sigma(j)\}$ and $I_2 = \{\sigma(j+1), \sigma(j+2), \dots, \sigma(i_1 - 1)]$, and if $\sigma(j) \in I_2$ then $I_1 = \{\sigma(1), \sigma(2), \dots, \sigma(j-1)\}$ and $I_2 = \{\sigma(j), \sigma(j+1), \dots, \sigma(i_1 - 1)]$. Hence, there are at most two ways to determine the sets I_1 and I_2 from σ. Similarly, there are at most two ways to determine each of the pair of sets I_{2i+1}, I_{2i+2}, where $1 \le i \le m$.

Thus, there exist at most 2^{m+1} different ways to obtain σ as described in Construction 17. ■

For two positive integers ℓ, r, where $r \le \ell$, the number of partitions of ℓ elements into r nonempty sets is denoted by $S(\ell, r)$ and is known as the Stirling number of the second kind [25, Ch. 13].

Lemma 22: Let n be an integer such that $n \equiv 0 \pmod 4$. Then the cardinality of the code $\mathcal{C}^{asym}_n$ satisfies

$$|\mathcal{C}^{asym}_n| \ge \sum_{r=3}^{n/2} \frac{1}{2} r! S\Big(n - 2\Big\lfloor \frac{r-1}{2} \Big\rfloor, r\Big) \Big\lfloor \frac{r-1}{2} \Big\rfloor!.$$

Proof: For every m, $1 \le m \le \frac{n}{4} - 1$, we compute a lower bound on the size of $\mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$. There are $r! S(n - 2m, r)$ choices for the partition $I_1, I_2, \dots, I_r$, where $r = 2m+1$ or $r = 2m+2$, and there are $m! \cdot 2^m$ choices for the permutation $\pi \in \mathcal{C}^{sym}_{2m,1}$. The expression

$$[(2m+1)! S(n - 2m, 2m+1) + (2m+2)! S(n - 2m, 2m+2)] m! 2^m$$

counts codewords in $\mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$ and by Lemma 21, each codeword in $\mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$ is counted at most 2^{m+1} times. Hence, the size of $\mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$ is at least

$$\begin{aligned}&[(2m+1)! S(n - 2m, 2m+1) \\ &\quad + (2m+2)! S(n - 2m, 2m+2)] \frac{m!}{2} \\ &= \sum_{r=2m+1}^{2m+2} \frac{1}{2} r! S\Big(n - 2\Big\lfloor \frac{r-1}{2} \Big\rfloor, r\Big) \Big\lfloor \frac{r-1}{2} \Big\rfloor!.\end{aligned}$$

By Lemma 19 it follows that the sets $\mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}$ and $\mathcal{C}_{2m'+1} \cup \mathcal{C}_{2m'+2}$ are disjoint if $m' \ne m$, and therefore

$$\begin{aligned}|\mathcal{C}^{asym}_n| &= \Big| \bigcup_{r=3}^{n/2} \mathcal{C}_r \Big| = \sum_{m=3}^{n/4-1} |\mathcal{C}_{2m+1} \cup \mathcal{C}_{2m+2}| \\ &\ge \sum_{r=3}^{n/2} \frac{1}{2} r! S\Big(n - 2\Big\lfloor \frac{r-1}{2} \Big\rfloor, r\Big) \Big\lfloor \frac{r-1}{2} \Big\rfloor!.\end{aligned}$$

■

In order to show that $\widetilde{C}(1) = 1$, we will need to use the following lower bound on the Stirling numbers of the second kind, which is taken from [19].

Lemma 23: For $1 \le r \le \ell$,

$$S(\ell, r) \ge \frac{1}{2}(r^2 + r + 2) r^{\ell - r - 1} - 1.$$

Finally, the next theorem, which is a direct result of Lemmas 22 and 23, highlights the main result of this section.

Theorem 24: For any function $k : \mathbb{N} \to \mathbb{N}$, $\widetilde{C}(k) = 1$.

Proof: Clearly, $\widetilde{C}(1) \le 1$. We will show that

$$\limsup_{n \to \infty} \frac{\log |B_{n,1}|}{\log n!} \ge 1,$$

by proving that for every $0 < \delta < \frac{1}{2}$,

$$\lim_{n \to \infty} \frac{\log |B_{4n,1}|}{\log (4n)!} > 1 - \delta.$$

Let δ be such that $0 < \delta < \frac{1}{2}$ and let $r = \lceil \delta 4n \rceil$. From Lemma 22 it follows that

$$\begin{aligned}|B_{4n,1}| &> \frac{1}{2} r! S\left(4n - 2\left\lfloor \frac{r-1}{2} \right\rfloor, r\right) \left\lfloor \frac{r-1}{2} \right\rfloor! \\ &\ge \frac{1}{2} r! S(4n - r, r) \left\lfloor \frac{r-1}{2} \right\rfloor!,\end{aligned}$$

and by Lemma 23

$$\begin{aligned}|B_{4n,1}| &> \frac{1}{4} r! r^{4n - 2r - 1} \left\lfloor \frac{r-1}{2} \right\rfloor! \\ &\ge \frac{1}{4} \lceil \delta 4n \rceil! (\delta 4n)^{4n(1-2\delta)-1} \left\lfloor \frac{\delta 4n - 1}{2} \right\rfloor! \\ &\ge \frac{1}{4} (\delta 4n/e)^{\delta 4n} (\delta 4n)^{4n(1-2\delta)-1} \left(\frac{\delta 4n - 2}{2e} \right)^{\frac{\delta 4n - 2}{2}} \\ &= c^n (\delta 4n)^{4n - \delta 2n},\end{aligned}$$

where c is some constant. It follows that

$$\lim_{n \to \infty} \frac{\log |B_{4n,1}|}{\log (4n)!} \ge \lim_{n \to \infty} \frac{\log c^n (\delta 4n)^{4n - \delta 2n}}{\log (4n)!} = 1 - \frac{\delta}{2} > 1 - \delta.$$

This shows that $\widetilde{C}(1) \ge 1$ and consequently it follows that $\widetilde{C}(k) = 1$, for every function $k : \mathbb{N} \to \mathbb{N}$. ■

V. The Capacity of Error-Correcting Constrained Codes

The two-neighbor constraint and the asymmetric two-neighbor constraint were proposed to combat errors that are caused by the inter-cell interference in flash memory cells. However, constrained codes should also be restricted to have error-correction capabilities, which is the topic of this section.

Given a permutation $\sigma = [\sigma(1), \sigma(2), \dots, \sigma(n)] \in S_n$, an ***adjacent transposition*** is an exchange of two adjacent elements $\sigma(i), \sigma(i+1)$, in σ, for some $1 \le i \le n-1$. The result of such an adjacent transposition is the permutation $[\sigma(1), \dots, \sigma(i-1), \sigma(i+1), \sigma(i), \sigma(i+2), \dots, \sigma(n)]$. The ***Kendall τ-distance*** [13] between two permutations $\sigma, \pi \in S_n$, denoted by $d_K(\sigma, \pi)$, is the minimum number of adjacent transpositions required to obtain the permutation π from the permutation σ.

Example 25: If $\sigma = [3,1,2,4]$ and $\pi = [1,3,4,2]$ then $d_K(\sigma,\pi) = 2$, since at least two adjacent transpositions are required to change the permutation σ to π: $[3,1,2,4] \to [1,3,2,4] \to [1,3,4,2]$.

For two permutations $\sigma, \pi \in S_n$ it is known [12], [14] that $d_K(\sigma,\pi)$ can be expressed as

$$d_K(\sigma,\pi) = |\{(i,j) : \sigma^{-1}(i) < \sigma^{-1}(j),\ \pi^{-1}(i) > \pi^{-1}(j)\}|. \tag{7}$$

For two permutations $\sigma, \pi \in S_n$ the ***inversion distance***, denoted by $d_I(\sigma,\pi)$, between σ and π is the Kendall τ-distance between their inverses, i.e.

$$d_I(\sigma,\pi) = d_K(\sigma^{-1},\pi^{-1}).$$

From (7) it follows that the Kendall τ-distance, and hence also the inversion distance, can only take integer values between 0 and $\binom{n}{2}$. For any permutation $\sigma \in S_n$, if $\pi = [\sigma(n), \sigma(n-1), \ldots, \sigma(1)]$ then $d_K(\sigma,\pi) = \binom{n}{2}$ and thus $d_I(\sigma^{-1},\pi^{-1}) = \binom{n}{2}$. Even though this distance was studied before, see e.g. [8], we are not aware of any formal name for this metric and thus call it here the inversion distance. In this section we study the capacity of the constraints in this paper combined with a requirement on the minimum inversion distance.

Remark 26: We study the inversion distance and not the Kendall τ-distance since, according to our representation of the cells ranking in a permutation, this metric fits better with the error behavior in flash memory cells. The motivation in studying codes in the Kendall τ-metric originated from the observation that cells with adjacent levels may interchange their rankings [12]. Therefore, codes in the Kendall τ-metric should be invoked over the inverse of the permutations. However, in order to study these codes with constrained codes, one should take the inversion distance applied for the permutations.

Let $E(n,k,d)$ be the maximum size of a code in $A_{n,k}$ with minimum inversion distance d. In this section we assume that k and d are two functions such that $k : \mathbb{N} \to \mathbb{N}$ and $d : \mathbb{N} \to \mathbb{Z}$ where $0 \le d(n) \le \binom{n}{2}$, for all $n \in \mathbb{N}$. Let us define the ***capacity*** of two-neighbor k-constrained codes with minimum inversion distance d by

$$C(k,d) = \limsup_{n\to\infty} \frac{\log E(n,k,d)}{\log n!}.$$

Let $\widetilde{E}(n,k,d)$ be the maximum size of a code in $B_{n,k}$ with minimum inversion distance d. Define the ***capacity*** of asymmetric two-neighbor k-constrained codes with minimum inversion distance d by

$$\widetilde{C}(k,d) = \limsup_{n\to\infty} \frac{\log \widetilde{E}(n,k,d)}{\log n!}.$$

Lastly, let $F(n,d)$ be the maximum size of a code in S_n with minimum inversion distance d. Define the ***capacity*** of codes in S_n with minimum inversion distance d by

$$C_{err}(d) = \limsup_{n\to\infty} \frac{\log F(n,d)}{\log n!}.$$

It was proved in [1] that for $d = \Theta(n^\delta)$,

$$C_{err}(d) = \begin{cases} 1, & 0 \le \delta \le 1, \\ 2-\delta, & 1 \le \delta \le 2. \end{cases}$$

We restrict our discussion only for functions k and d such that $k = \Theta(n^\epsilon)$ and $d = \Theta(n^\delta)$, for some ϵ and δ, $0 \le \epsilon \le 1$ and $0 \le \delta \le 2$. In particular, we will show that $\widetilde{C}(k,d) = C_{err}(d)$, for every k. For the computation of $C(k,d)$ we distinguish between three cases:

1) $0 \le \epsilon \le 1$ and $0 \le \delta \le 1$.
2) $0 \le \epsilon \le 1$ and $1 < \delta \le 1+\epsilon$.
3) $0 \le \epsilon < 1$ and $1+\epsilon < \delta \le 2$.

The computation of the capacity for these three cases is presented in the three Subsections V-A, V-B, and V-C. The computation of the capacity for asymmetric two-neighbor k-constrained error-correcting codes is presented in Subsection V-D. Similarly to previous sections, the computation of both capacities $C(k,d)$ and $\widetilde{C}(k,d)$ will be based on finding upper and lower bounds on the size of $E(n,k,d)$ and $\widetilde{E}(n,k,d)$, respectively.

A. The Case $0 \le \epsilon \le 1$ and $0 \le \delta \le 1$

In this subsection we will compute the capacity $C(k,d)$, where $k = \Theta(n^\epsilon)$, $d = \Theta(n^\delta)$, and $0 \le \epsilon, \delta \le 1$. To this end, we will need some definitions and notations.

For $\sigma \in S_n$, the ***ball*** in S_n of radius r centered at σ is defined by

$$\mathcal{B}_I(n,\sigma,r) \overset{\text{def}}{=} \{\pi \in S_n \ : \ d_I(\sigma,\pi) \le r\}.$$

The size of the ball $\mathcal{B}_I(n,\sigma,r)$ does not depend on σ and thus we denote it by $b_I(n,r)$. For $\sigma \in A_{n,k}$, the ***ball*** in $A_{n,k}$ of radius r centered at σ is defined by

$$\mathcal{B}_I(A_{n,k},\sigma,r) \overset{\text{def}}{=} \{\pi \in A_{n,k} \ : \ d_I(\sigma,\pi) \le r\}.$$

A code in $A_{n,k}$ with minimum inversion distance d can be constructed by a greedy approach which leads to the following Gilbert-Varshamov type of lower bound.

Lemma 27: For every $1 \le k$ and $1 \le d \le \binom{n}{2}$, the following lower bound on $E(n,k,d)$ holds

$$E(n,k,d) \ge \frac{|A_{n,k}|}{b_I(n,d-1)}.$$

The next theorem is a combination of results from [1], [16], and [17].

Theorem 28: Let $r = \Theta(n^\delta)$, where $0 \le \delta \le 2$. Then there exist constants c_1 and c_2 such that

$$b_I(n,r) \le \begin{cases} {c_1}^n, & 0 \le \delta \le 1, \\ (c_2 n^{\delta-1})^n, & 1 < \delta \le 2. \end{cases}$$

We are now in a position to compute the capacity $C(k,d)$ for the first case.

Theorem 29: If $k = \Theta(n^\epsilon)$ and $d = \Theta(n^\delta)$, where $0 \le \epsilon, \delta \le 1$, then $C(k,d) = \frac{1}{2} + \frac{\epsilon}{2}$.

Proof: Since $E(n,k,d) \subseteq A_{n,k}$ it follows that

$$C(k,d) = \limsup_{n\to\infty} \frac{\log E(n,k,d)}{\log n!} \le \limsup_{n\to\infty} \frac{\log |A_{n,k}|}{\log n!} = C(k),$$

and hence from Corollary 16, $C(k,d) \le C(k) = \frac{1}{2} + \frac{\epsilon}{2}$.

By Lemma 27 and Theorem 28 there exists a constant c such that

$$\frac{\log E(n,k,d)}{\log n!} \ge \frac{\log |A_{n,k}|}{\log n!} - \frac{\log c^n}{\log n!}.$$

Since $\lim_{n\to\infty} \log c^n / \log n! = 0$ it follows that

$$C(k,d) = \limsup_{n\to\infty} \frac{\log E(n,k,d)}{\log n!} \ge \limsup_{n\to\infty} \frac{\log |A_{n,k}|}{\log n!} = C(k).$$

Then, $C(k,d) \ge C(k) = \frac{1}{2} + \frac{\epsilon}{2}$, and thus $C(k,d) = \frac{1}{2} + \frac{\epsilon}{2}$. ■

B. The Case $0 \le \epsilon \le 1$ and $1 < \delta \le 1+\epsilon$

In this subsection we will compute the capacity $C(k,d)$, where $k = \Theta(n^\epsilon)$, $d = \Theta(n^\delta)$, $0 \le \epsilon \le 1$ and $1 < \delta \le 1+\epsilon$. First, let us introduce some more tools that will be used in solving this case.

Let $H_n = \{1, 2, \ldots, n\}^n$. The definition of the two-neighbor k-constraint can be trivially extended to H_n. A vector $\mathbf{x} \in H_n$ satisfies the two-neighbor k-constraint if $|x_i - x_{i-1}| \le k$ or $|x_{i+1} - x_i| \le k$, for all $2 \le i \le n-1$. Let $\vec{\mathcal{A}}_{n,k}$ be the set of all vectors of H_n that satisfy the two-neighbor k-constraint. The next lemma can be proved by following the same lines that were used to prove Lemma 14, and thus its proof is omitted.

Lemma 30: If $1 \le k$ then

$$|\vec{\mathcal{A}}_{n,k}| \le 4^{n-1}(k+1)^{\frac{n}{2}} n^{\frac{n}{2}+1}.$$

For $\mathbf{x}, \mathbf{y} \in H_n$, the ***Manhattan distance*** between $\mathbf{x}$ and $\mathbf{y}$, $d_M(\mathbf{x}, \mathbf{y})$, is defined as

$$d_M(\mathbf{x}, \mathbf{y}) \overset{\text{def}}{=} \sum_{i=1}^{n} |x_i - y_i|.$$

The next lemma was proved in [8].

Lemma 31: For every $\sigma, \pi \in S_n$,

$$\frac{1}{2} d_M(\sigma, \pi) \le d_I(\sigma, \pi) \le d_M(\sigma, \pi).$$

For a subset $S \subseteq H_n$ and $\mathbf{x} \in S$, the Manhattan ***ball*** in S of radius r centered at $\mathbf{x}$ is defined by

$$\mathcal{B}_M(S, \mathbf{x}, r) \overset{\text{def}}{=} \{\mathbf{y} \in S \ : \ d_M(\mathbf{x}, \mathbf{y}) \le r\}.$$

Combining the previous result along with the sphere packing upper bound provides us with the following lemma.

Lemma 32: For every $1 \le k$ and $1 \le d \le \binom{n}{2}$,

$$E(n,k,d) \le \frac{|\vec{\mathcal{A}}_{n,k}|}{\min_{\mathbf{x} \in \vec{\mathcal{A}}_{n,k}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, \lfloor \frac{d-1}{2} \rfloor)|\}}.$$

Proof: From Lemma 31 it follows that every code in $A_{n,k}$ with minimum inversion distance d is also a code in $\vec{\mathcal{A}}_{n,k}$ with minimum Manhattan distance d. Hence, by the sphere packing bound for codes in $\vec{\mathcal{A}}_{n,k}$ the following upper bound holds

$$E(n,k,d) \le \frac{|\vec{\mathcal{A}}_{n,k}|}{\min_{\mathbf{x} \in \vec{\mathcal{A}}_{n,k}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, \lfloor \frac{d-1}{2} \rfloor)|\}}.$$

■

In order to apply the upper bound on $E(n,k,d)$ from Lemma 32, we need a lower bound on the size of a Manhattan ball in $\vec{\mathcal{A}}_{n,k}$. Next, we will present some tools that will be useful for finding such a lower bound.

For every three positive integers m, ℓ, t, let $Q_{m,\ell,t}$ be the set defined by

$$Q(m,\ell,t) = \left\{ (y_1, \ldots, y_m) \in \mathbb{Z}^m \ : \ \begin{array}{c} \sum_{i=1}^{m} y_i \le t, \\ \forall 1 \le i \le m, \ 0 \le y_i \le \ell \end{array} \right\}.$$

The following lemma was proved in [20].

Lemma 33: If $\ell = \Theta(m^\epsilon)$ and $t = \Theta(m^\delta)$, where $\delta < 1+\epsilon$, then

$$|Q(m,\ell,t)| \ge \left(\frac{t}{m}\right)^m,$$

for sufficiently large m.

For simplicity, we assume in the rest of this section that all integers divisions result in integer numbers. We note that this assumption does not affect the forthcoming results, since otherwise small modifications can be applied.

For positive integers n, k and r such that $2k < \frac{n}{2}$, let $D_1(n,k,r) = Q\left(\frac{n}{3}, \frac{n}{2} - 2k, \frac{3r}{20}\right)$ and $D_2(n,k,r) = Q\left(\frac{2n}{3}, \frac{k}{2}, \frac{r}{8}\right)$, i.e.

$$D_1(n,k,r) = \left\{ (y_1, \ldots, y_{\frac{n}{3}}) \in \mathbb{Z}^{\frac{n}{3}} \ : \ \begin{array}{c} \sum_{i=1}^{\frac{n}{3}} y_i \le \frac{3r}{20}, \\ 0 \le y_i \le \frac{n}{2} - 2k, \\ \forall 1 \le i \le \frac{n}{3} \end{array} \right\}$$

and

$$D_2(n,k,r) = \left\{ (z_1, \ldots, z_{\frac{2n}{3}}) \in \mathbb{Z}^{\frac{2n}{3}} \ : \ \begin{array}{c} \sum_{i=1}^{\frac{2n}{3}} z_i \le \frac{r}{8}, \\ 0 \le z_i \le \frac{k}{2}, \\ \forall 1 \le i \le \frac{2n}{3} \end{array} \right\}.$$

Lemma 34: If $k = \Theta(n^\epsilon)$ and $r = \Theta(n^\delta)$, where $0 \le \epsilon < 1$ and $1 < \delta \le 1+\epsilon$, then there exists a constant c such that for n large enough

$$|D_1(n,k,r)| \cdot |D_2(n,k,r)| \ge \left(cn^{\delta-1}\right)^n.$$

Proof: Since $\epsilon < 1$ it follows that $\frac{n}{2} - 2k = \Theta(n)$ and since $\delta < 2$ it follows from Lemma 33 that

$$|D_1(n,k,r)| = \left| Q\left(\frac{n}{3}, \frac{n}{2} - 2k, \frac{3r}{20}\right) \right| \ge \left(\frac{\frac{3r}{20}}{\frac{n}{3}}\right)^{\frac{n}{3}} \ge \left(c_1 n^{\delta-1}\right)^{\frac{n}{3}},$$

for some constant c_1.

If $\delta < 1+\epsilon$ then by Lemma 33 it follows that

$$|D_2(n,k,r)| \ge \left(\frac{\frac{r}{8}}{\frac{2n}{3}}\right)^{\frac{2n}{3}} \ge \left(c_2 n^{\delta-1}\right)^{\frac{2n}{3}},$$

for some constant c_2. Thus, there exists a constant c such that

$$|D_1(n,k,r)| \cdot |D_2(n,k,r)| \ge \left(cn^{\delta-1}\right)^n.$$

If $\delta = 1+\epsilon$ then there exists some constant $\widetilde{c}_1$ such that $\frac{kn}{\widetilde{c}_1} \le \frac{r}{8}$, for sufficiently large n. Let $\ell = \min\left\{\frac{k}{\widetilde{c}_1}, \frac{k}{2}\right\}$, then $[\ell]^{\frac{2n}{3}} \subseteq D_2(n,k,r)$, and therefore there exists some constant $\widetilde{c}_2$ such that $|D_2(n,k,r)| \ge (\widetilde{c}_2 n^\epsilon)^{\frac{2n}{3}}$.

Thus, there exists a constant c such that

$$|D_1(n,k,r)|\cdot|D_2(n,k,r)| \geq c^n \left(n^{\delta-1}\right)^{\frac{n}{3}} \left(n^{\epsilon}\right)^{\frac{2n}{3}} =^{(a)} c^n \left(n^{\frac{\delta-1}{3}+\frac{2(\delta-1)}{3}}\right)^n = \left(cn^{\delta-1}\right)^n,$$

where equality (a) follows from $\epsilon = \delta - 1$. ∎

Note that Lemma 34 follows from the fact that $D_1(n,k,r) = Q(m_1, \ell_1, t_1)$ and $D_2(n,k,r) = Q(m_2, \ell_2, t_2)$, where $m_1 + m_2 = n$, $\ell_1, \ell_2 = \Omega(k)$, and $t_1, t_2 = \Omega(r)$. The purpose of the sets $D_1(n,k,r)$ and $D_2(n,k,r)$ is to establish a lower bound on $|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)|$ in terms of $|D_1(n,k,r)| \cdot |D_2(n,k,r)|$ as stated in the next lemma, which is proved in Appendix A. This is accomplished by introducing a mapping from $D_1(n,k,r) \times D_2(n,k,r)$ to $\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)$. For the design of this mapping it is crucial that m_1 and m_2 take the values $\frac{n}{3}$, and $\frac{2n}{3}$, respectively, whereas the remaining parameters of $D_1(n,k,r)$ and $D_2(n,k,r)$ can be chosen in many ways.

Lemma 35: For every three positive integers n, k, r such that $2k < \frac{n}{2}$, and for all $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$,

$$|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)| \geq \frac{|D_1(n,k,r)| \cdot |D_2(n,k,r)|}{4^{\frac{2n}{3}}}.$$

Corollary 36: Let $k = \Theta(n^{\epsilon})$ and $r = \Theta(n^{\delta})$, where $0 \leq \epsilon < 1$ and $1 < \delta \leq 1+\epsilon$. Then there exists a constant c such that

$$\min_{\mathbf{x}\in\vec{\mathcal{A}}_{n,k}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)|\} \geq \left(cn^{\delta-1}\right)^n.$$

Proof: By Lemmas 34 and 35 it follows that there exists a constant c such that

$$|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)| \geq \left(cn^{\delta-1}\right)^n,$$

for all $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$.

Thus,

$$\min_{\mathbf{x}\in\vec{\mathcal{A}}_{n,k}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)|\} \geq \left(cn^{\delta-1}\right)^n.$$

∎

We are now ready to compute the capacity for the second case.

Theorem 37: If $k = \Theta(n^{\epsilon})$ and $d = \Theta(n^{\delta})$, where $0 \leq \epsilon \leq 1$ and $1 < \delta \leq 1+\epsilon$, then

$$C(k,d) = \frac{3}{2} + \frac{\epsilon}{2} - \delta.$$

Proof: By Lemma 27 and Theorem 28 it follows that there exists a constant c_1 such that

$$\frac{\log E(n,k,d)}{\log n!} \geq \frac{\log |A_{n,k}|}{\log n!} - \frac{\log c_1^n n^{(\delta-1)n}}{\log n!}.$$

Since $\lim_{n\to\infty} \log c_1^n n^{(\delta-1)n} / \log n! = \delta - 1$ and by Corollary 16 it follows that

$$C(k,d) \geq \frac{1}{2} + \frac{\epsilon}{2} + 1 - \delta = \frac{3}{2} + \frac{\epsilon}{2} - \delta.$$

If $\epsilon = 1$ then $C(k,d) \geq 2-\delta$. On the other hand $C(k,d) \leq C_{err}(d) = 2 - \delta$, and thus $C(k,d) = 2-\delta$.

If $\epsilon < 1$ then by the upper bound from Lemma 32 and by Corollary 36 it follows that

$$\frac{\log E(n,k,d)}{\log n!} \leq \frac{\log |\vec{\mathcal{A}}_{n,k}|}{\log n!} - \frac{\log \left(c_2 n^{\delta-1}\right)^n}{\log n!},$$

for some constant c_2. By Lemma 30 it follows that

$$|\vec{\mathcal{A}}_{n,k}| \leq 4^{n-1}(k+1)^{\frac{n}{2}} n^{\frac{n}{2}+1},$$

and hence

$$\limsup_{n\to\infty} \frac{\log |\vec{\mathcal{A}}_{n,k}|}{\log n!} \leq \frac{1}{2} + \frac{\epsilon}{2}.$$

Therefore,

$$C(k,d) \leq \frac{1}{2} + \frac{\epsilon}{2} + 1 - \delta.$$

Thus, $C(k,d) = \frac{3}{2} + \frac{\epsilon}{2} - \delta$. ∎

C. The Case $0 \leq \epsilon < 1$ and $1+\epsilon < \delta \leq 2$

The goal of this subsection is to compute the capacity $C(k,d)$, where $k = \Theta(n^{\epsilon})$, $d = \Theta(n^{\delta})$, $0 \leq \epsilon < 1$ and $1+\epsilon < \delta \leq 2$.

Lemma 38: For every $1 \leq k$ and $1 \leq d \leq \binom{n}{2}$,

$$E(n,k,d) \geq \frac{|A_{n,k}|}{\max_{\mathbf{x}\in\vec{\mathcal{A}}_{n,k}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, 2d-1)|\}}.$$

Proof: From Lemma 31 it follows that every code in $A_{n,k}$ with minimum Manhattan distance $2d$ is also a code in $A_{n,k}$ with minimum inversion distance d. Hence,

$$E(n,k,d) \geq \frac{|A_{n,k}|}{\max_{\mathbf{x}\in A_{n,k}} \{|\mathcal{B}_M(A_{n,k}, \mathbf{x}, 2d-1)|\}},$$

and since

$$\max_{\mathbf{x}\in A_{n,k}} \{|\mathcal{B}_M(A_{n,k}, \mathbf{x}, 2d-1)|\} \leq \max_{\mathbf{x}\in\vec{\mathcal{A}}_{n,k}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, 2d-1)|\},$$

we get

$$E(n,k,d) \geq \frac{|A_{n,k}|}{\max_{\mathbf{x}\in\vec{\mathcal{A}}_{n,k}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, 2d-1)|\}}.$$

∎

In order to apply the lower bound from Lemma 38 we state in the following lemma an upper on the size of a Manhattan ball in $\vec{\mathcal{A}}_{n,k}$. The proof of this lemma appears in Appendix B.

Lemma 39: Let $k = \Theta(n^{\epsilon})$ and $r = \Theta(n^{\delta})$, where $0 \leq \epsilon < 1$ and $1 \leq \delta < 2$. Then, there exists a constant c such that for n large enough

$$\max_{\mathbf{x}\in\vec{\mathcal{A}}_{n,k}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)|\} \leq c^n n^{(\delta-1+\epsilon)\frac{n}{2}}.$$

Let

$$\vec{\mathcal{A}}_{n,k,alt} \stackrel{\text{def}}{=} \{\mathbf{x} \in H_n \ : \ |x_{2i} - x_{2i-1}| \leq k, \ \text{for all } 1 \leq i \leq \frac{n}{2}\}.$$

Note that if $\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}$ then $\mathbf{x}$ satisfies the two-neighbor k-constraint, and therefore $\vec{\mathcal{A}}_{n,k,alt} \subseteq \vec{\mathcal{A}}_{n,k}$. We will show how to find an upper bound on $E(n,k,d)$ by using a sphere packing bound for codes in $\vec{\mathcal{A}}_{n,k,alt}$. But, first we need more definitions.

Define the mapping $\mu : A_{n,k} \to \{0,1\}^{n-1}$, where for every $\sigma \in A_{n,k}$, $\mu(\sigma) = \mathbf{y}$, $\mathbf{y} = (y_1, y_2, \ldots, y_{n-1})$, is defined as follows. For every $1 \le i \le n-1$

$$y_i = \begin{cases} 1, & |y_{i+1} - y_i| \le k, \\ 0, & |y_{i+1} - y_i| > k. \end{cases}$$

For every $\mathbf{y} \in \{0,1\}^{n-1}$, define

$$A_{n,k,\mathbf{y}} \stackrel{\text{def}}{=} \{\sigma \in A_{n,k} \ : \ \mu(\sigma) = \mathbf{y}\}$$

and let $E(n,k,d,\mathbf{y})$ be the maximum size of a code in $A_{n,k,\mathbf{y}}$ with minimum inversion distance d.

Lemma 40: For every $1 \le k$ and $1 \le d \le \binom{n}{2}$,

$$E(n,k,d) \le \sum_{\mathbf{y} \in \{0,1\}^{n-1}} E(n,k,d,\mathbf{y}).$$

Proof: Let $\mathcal{C} \subseteq A_{n,k}$ be a code with minimum inversion distance d and of size $E(n,k,d)$. For every $\mathbf{y} \in \{0,1\}^{n-1}$, let

$$\mathcal{C}_{\mathbf{y}} = \mathcal{C} \cap A_{n,k,\mathbf{y}}.$$

Clearly, $\mathcal{C}_{\mathbf{y}}$ is a code in $A_{n,k,\mathbf{y}}$ with minimum inversion distance at least d. Then,

$$E(n,k,d) = |\mathcal{C}| = \sum_{\mathbf{y} \in \{0,1\}^{n-1}} |\mathcal{C}_{\mathbf{y}}| \le \sum_{\mathbf{y} \in \{0,1\}^{n-1}} E(n,k,d,\mathbf{y}).$$

■

The proof of the next lemma appears in Appendix C.

Lemma 41: If $\mathbf{y} \in \{0,1\}^{n-1}$, $1 \le k$, and $1 \le d \le \binom{n}{2}$, such that $d > (2k/3+2)n$, then

$$E(n,k,d,\mathbf{y}) \le \frac{2^n n^2 |\vec{\mathcal{A}}_{n,k,alt}|}{\min_{\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, r)|\}},$$

where $r = \frac{\frac{d}{2} - \left(\frac{k}{3}+1\right)n - 1}{2}$.

Combining Lemmas 40 and 41 we conclude with the following corollary.

Corollary 42: For every $1 \le k$ and $1 \le d \le \binom{n}{2}$, such that $d > (2k/3+2)n$,

$$E(n,k,d) \le \frac{4^n n^2 |\vec{\mathcal{A}}_{n,k,alt}|}{\min_{\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, r)|\}},$$

where $r = \frac{\frac{d}{2} - \left(\frac{k}{3}+1\right)n - 1}{2}$.

In order to apply the upper bound from Corollary 42 we need a lower bound on the size of a Manhattan ball in $\vec{\mathcal{A}}_{n,k,alt}$. This is accomplished by using similar methods to those that were used in Subsection V-B, to obtain a lower bound on the size of a Manhattan ball in $\vec{\mathcal{A}}_{n,k}$.

Let n, k, r be integers such that $k < \frac{n}{2}$ and let $\widetilde{D}_1(n,k,r) = Q\left(\frac{n}{2}, \frac{n}{2} - k, \frac{r}{4}\right)$ and $\widetilde{D}_2(n,k,r) = Q\left(\frac{n}{2}, k, \frac{r}{2}\right)$, i.e.

$$\widetilde{D}_1(n,k,r) = \left\{ (y_1, y_2, \ldots, y_{\frac{n}{2}}) \in \mathbb{Z}^{\frac{n}{2}} : \begin{array}{c} \sum_{i=1}^{\frac{n}{2}} y_i \le \frac{r}{4}, \\ 0 \le y_i \le \frac{n}{2} - k, \\ \forall 1 \le i \le \frac{n}{2}, \end{array} \right\},$$

and

$$\widetilde{D}_2(n,k,r) = \left\{ (z_1, z_2, \ldots, z_{\frac{n}{2}}) \in \mathbb{Z}^{\frac{n}{2}} : \begin{array}{c} \sum_{i=1}^{\frac{n}{2}} z_i \le \frac{r}{2}, \\ 0 \le z_i \le k, \\ \forall 1 \le i \le \frac{n}{2} \end{array} \right\}.$$

Lemma 43: If $k = \Theta(n^\epsilon)$ and $r = \Theta(n^\delta)$, where $0 \le \epsilon < 1$ and $1 + \epsilon < \delta < 2$, then there exists a constant c such that for n large enough

$$|\widetilde{D}_1(n,k,r)| \cdot |\widetilde{D}_2(n,k,r)| \ge \left(cn^{\delta-1+\epsilon}\right)^{\frac{n}{2}}.$$

Proof: Since $\epsilon < 1$ it follows that $\frac{n}{2} - k = \Theta(n)$ and since $\delta < 2$ it follows from Lemma 33 that

$$|\widetilde{D}_1(n,k,r)| = \left|Q\left(\frac{n}{2}, \frac{n}{2} - k, \frac{r}{4}\right)\right| \ge \left(\frac{\frac{r}{4}}{\frac{n}{2}}\right)^{\frac{n}{2}} \ge \left(c_1 n^{\delta-1}\right)^{\frac{n}{2}},$$

for some constant c_1.

Since $1 + \epsilon < \delta < 2$, it follows that $kn \le \frac{r}{4}$, for sufficiently large n. Then $[k]^m \subseteq \widetilde{D}_2(n,k,r)$, and therefore there exists some constant c_2 such that $|\widetilde{D}_2(n,k,r)| \ge (c_2 n^\epsilon)^{\frac{n}{2}}$.

Thus, there exists a constant c such that

$$|\widetilde{D}_1(n,k,r) \times \widetilde{D}_2(n,k,r)| \ge \left(cn^{\delta-1}n^\epsilon\right)^{\frac{n}{2}} = \left(cn^{\delta-1+\epsilon}\right)^{\frac{n}{2}}.$$

■

The proof of the following lemma appears in Appendix D.

Lemma 44: For every three positive integers n, k, r such that $2k < \frac{n}{2}$, and for all $\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}$

$$|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, r)| \ge \frac{|\widetilde{D}_1(n,k,r)| \cdot |\widetilde{D}_2(n,k,r)|}{4^{\frac{n}{2}}}.$$

As an immediate consequence of Lemmas 43 and 44 we derive the following corollary.

Corollary 45: Let $k = \Theta(n^\epsilon)$ and $r = \Theta(n^\delta)$, where $0 \le \epsilon < 1$ and $1 + \epsilon < \delta < 2$. Then there exists a constant c such that

$$\min_{\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, r)|\} \ge \left(cn^{\delta-1+\epsilon}\right)^{\frac{n}{2}}.$$

Theorem 46: If $k = \Theta(n^\epsilon)$ and $d = \Theta(n^\delta)$, where $0 \le \epsilon < 1$ and $1 + \epsilon < \delta \le 2$, then

$$C(k,d) = 1 - \frac{\delta}{2}.$$

Proof: For $\delta < 2$ it follows from Lemmas 38 and 39 that

$$\frac{\log E(n,k,d)}{\log n!} \ge \frac{\log |A_{n,k}|}{\log n!} - \frac{\log(c_1^n n^{(\delta-1+\epsilon)\frac{n}{2}})}{\log n!},$$

for some constant c_1. Thus,

$$C(k,d) \ge \frac{1}{2} + \frac{\epsilon}{2} + \frac{1}{2} - \frac{\delta}{2} - \frac{\epsilon}{2} = 1 - \frac{\delta}{2}.$$

Note that since $\delta > 1 + \epsilon$, then for n large enough, $d > (2k/3+2)n$. Hence we can apply Corollary 42 and together with Corollary 45 we get that

$$\frac{\log E(n,k,d)}{\log n!} \le \frac{\log 4^n n^2 |\vec{\mathcal{A}}_{n,k,alt}|}{\log n!} - \frac{\log \left(c_2 n^{\delta-1+\epsilon}\right)^{\frac{n}{2}}}{\log n!},$$

for some constant c_2. Since $|\vec{\mathcal{A}}_{n,k,alt}| \le |\vec{\mathcal{A}}_{n,k}|$ and by Lemma 30 it follows that

$$C(k,d) \le \frac{1}{2} + \frac{\epsilon}{2} + \frac{1}{2} - \frac{\delta}{2} - \frac{\epsilon}{2} = 1 - \frac{\delta}{2}.$$

Lastly, for $\delta = 2$, recall that the capacity of a code in S_n with minimum inversion distance $d = \Theta(n^2)$, $C_{err}(d)$, is equal to 0 as was proved in [1]. Clearly, $0 \le C(k,d) \le C_{err}(d)$, and thus $C(k,d) = 0$.

We conclude that if $k = \Theta(n^{\epsilon})$ and $d = \Theta(n^{\delta})$, where $0 \le \epsilon \le 1$ and $1+\epsilon < \delta \le 2$, then $C(k,d) = 1 - \frac{\delta}{2}$. ■

For conclusion, Theorems 29, 37, and 46 are summarized in the following corollary.

Corollary 47: If $k = \Theta(n^{\epsilon})$ and $d = \Theta(n^{\delta})$, where $0 \le \epsilon \le 1$ and $0 \le \delta \le 2$, then

$$C(k,d) = \begin{cases} \frac{1}{2} + \frac{\epsilon}{2}, & 0 \le \delta \le 1, \\ \frac{3}{2} + \frac{\epsilon}{2} - \delta, & 1 < \delta \le 1+\epsilon, \\ 1 - \frac{\delta}{2}, & 1+\epsilon < \delta \le 2. \end{cases}$$

D. Computation of $\widetilde{C}(k,d)$

To compute the capacity of asymmetric two-neighbor k-constrained codes, $\widetilde{C}(k,d)$, we need the following Gilbert-Varshamov type of lower bound on $\widetilde{E}(n,k,d)$.

Lemma 48: For every $1 \le k$ and $1 \le d \le \binom{n}{2}$,

$$\widetilde{E}(n,k,d) \ge \frac{|B_{n,k}|}{b_I(n,d-1)}.$$

Theorem 49: If $k = \Theta(n^{\epsilon})$ and $d = \Theta(n^{\delta})$, where $0 \le \epsilon \le 1$ and $0 \le \delta \le 2$, then

$$\widetilde{C}(k,d) = C_{err}(d) = \begin{cases} 1, & 0 \le \delta \le 1, \\ 2-\delta, & 1 \le \delta \le 2. \end{cases}$$

Proof: Clearly,

$$\widetilde{C}(k,d) \le C_{err}(d) = \begin{cases} 1, & 0 \le \delta \le 1, \\ 2-\delta, & 1 \le \delta \le 2. \end{cases}$$

From Lemma 48 it follows that

$$\frac{\log \widetilde{E}(n,k,d)}{\log n!} \ge \frac{\log |B_{n,k}|}{n!} - \frac{\log b_I(n,d-1)}{\log n!},$$

and by Theorem 24

$$\widetilde{C}(k,d) \ge 1 - \limsup_{n\to\infty} \frac{\log b_I(n,d-1)}{\log n!}. \quad (8)$$

For $0 \le \delta \le 1$, it follows from (8) and Theorem 28 that

$$\widetilde{C}(k,d) \ge 1 - \limsup_{n\to\infty} \frac{\log c_1^n}{\log n!} = 1,$$

where c_1 is some constant.

For $1 < \delta \le 2$, it follows from (8) and Theorem 28 that

$$\widetilde{C}(k,d) \ge 1 - \limsup_{n\to\infty} \frac{\log (c_2 n^{\delta-1})^n}{\log n!} = 2-\delta,$$

where c_2 is some constant.

Thus,

$$\widetilde{C}(k,d) = \begin{cases} 1, & 0 \le \delta \le 1, \\ 2-\delta, & 1 \le \delta \le 2. \end{cases}$$

■

VI. Conclusions

In this paper we studied constrained codes for permutations. The motivation for these constraints originates from the inter-cell interference phenomenon in flash memories, where cells with high charge can affect neighbor cells with low charge. We focused on two families of constraints, namely, the two-neighbor k-constraint and the asymmetric two-neighbor k-constraint. For each constraint, we first calculated the capacity of the constraint when k is of the form $k = \Theta(n^{\epsilon})$. Then, we continued to study the capacity of each constraint when requiring the constrained codes to also have a minimum inversion distance d, given by $d = \Theta(n^{\delta})$.

Appendix A

The purpose of this appendix is to prove Lemma 35 from Section V, i.e. for every three positive integers n,k,r such that $2k < \frac{n}{2}$, and for all $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$,

$$|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k},\mathbf{x},r)| \ge \frac{|D_1(n,k,r)| \cdot |D_2(n,k,r)|}{4^{\frac{2n}{3}}}.$$

Recall, that $D_1(n,k,r) = Q\left(\frac{n}{3}, \frac{n}{2}-2k, \frac{3r}{20}\right)$ and $D_2(n,k,r) = Q\left(\frac{2n}{3}, \frac{k}{2}, \frac{r}{8}\right)$, i.e.

$$D_1(n,k,r) = \left\{ (y_1,\ldots,y_{\frac{n}{3}}) \in \mathbb{Z}^{\frac{n}{3}} : \begin{array}{c} \sum_{i=1}^{\frac{n}{3}} y_i \le \frac{3r}{20}, \\ 0 \le y_i \le \frac{n}{2}-2k, \\ \forall 1 \le i \le \frac{n}{3} \end{array} \right\}$$

and

$$D_2(n,k,r) = \left\{ (z_1,\ldots,z_{\frac{2n}{3}}) \in \mathbb{Z}^{\frac{2n}{3}} : \begin{array}{c} \sum_{i=1}^{\frac{2n}{3}} z_i \le \frac{r}{8}, \\ 0 \le z_i \le \frac{k}{2}, \\ \forall 1 \le i \le \frac{2n}{3} \end{array} \right\}.$$

To accomplished our goal we will define a mapping $\psi_{\mathbf{x}} : D_1(n,k,r) \times D_2(n,k,r) \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k},\mathbf{x},r)$, for every $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$, such that

$$|\{(\mathbf{y},\mathbf{z}) \in D_1(n,k,r) \times D_2(n,k,r) \; : \; \psi_{\mathbf{x}}(\mathbf{y},\mathbf{z}) = \mathbf{u}\}| \le 4^{\frac{2n}{3}},$$

for every $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k},\mathbf{x},r)$. For ease of notation we denote the set $D_1(n,k,r)$ by D_1 and the set $D_2(n,k,r)$ by D_2. The mapping $\psi_{\mathbf{x}}$ will be defined by two other mappings $\rho_{\mathbf{x}}$ and $\xi_{\mathbf{w}}$, where $\rho_{\mathbf{x}} : D_1 \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k},\mathbf{x},\frac{3r}{4})$ and $\xi_{\mathbf{w}} : D_2 \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k},\mathbf{w},\frac{r}{4})$, for all $\mathbf{w} \in \vec{\mathcal{A}}_{n,k}$. We define the sets I_1, I_2, I_3 as follows:

$$I_1 = \{i \in [n] \; : \; i \equiv 1 \text{ (mod3)}\},$$
$$I_2 = \{i \in [n] \; : \; i \equiv 2 \text{ (mod3)}\},$$
$$I_3 = \{i \in [n] \; : \; i \equiv 3 \text{ (mod3)}\}.$$

The goal of the mapping $\rho_{\mathbf{x}}$ is to invoke high changes on third of the entries in $\mathbf{x}$ that are specified by the set of indices I_2, and accordingly change the remaining entries to preserve the two-neighbor k-constraint. Then, the mapping $\xi_{\mathbf{w}}$ changes only the entries of the indices from $I_1 \cup I_3$ while again preserving the two-neighbor k-constraint. Clearly, the definition of $\rho_{\mathbf{x}}$ (of $\xi_{\mathbf{w}}$, respectively) on the ith entry, for $i \in I_1 \cup I_3$, should depend on whether $|x_i - x_{i-1}| \le k$ or $|x_{i+1} - x_i| \le k$ (on whether $|w_i - w_{i-1}| \le k$ or $|w_{i+1} - w_i| \le k$, respectively).

Therefore, we will partition the set $I_1 \cup I_3$ into three sets $J(\mathbf{x})$, $K(\mathbf{x})$, and $L(\mathbf{x})$, according to the differences between neighboring entries in $\mathbf{x}$, and define $\rho_{\mathbf{x}}$ for each of these sets to preserve the two-neighbor k-constraint. Similarly, we will partition the set $I_1 \cup I_3$ again into three sets $J(\mathbf{w})$, $K(\mathbf{w})$, and $L(\mathbf{w})$, according to the differences between neighboring entries in $\mathbf{w}$, and define $\xi_{\mathbf{w}}$ for each of these sets to preserve the two-neighbor k-constraint.

For every $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, define $J(\mathbf{x}) \subseteq I_1 \cup I_3$ by

$$J(\mathbf{x}) = \left\{ i \in I_1 \cup I_3 \; : \begin{array}{c} i-1 \in I_2 \wedge |x_i - x_{i-1}| \leq k \\ \text{or} \\ i+1 \in I_2 \wedge |x_{i+1} - x_i| \leq k \end{array} \right\}$$

and let

$$J_1(\mathbf{x}) = J(\mathbf{x}) \cap I_1 \quad \text{and } J_3(\mathbf{x}) = J(\mathbf{x}) \cap I_3.$$

Note, that if $i \in J_1(\mathbf{x})$ then $i+1 \in I_2$ and $|x_{i+1} - x_i| \leq k$. Similarly, if $i \in J_3(\mathbf{x})$ then $i-1 \in I_2$ and $|x_i - x_{i-1}| \leq k$. For every $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$, define $K(\mathbf{x}) \subseteq (I_1 \cup I_3) \setminus J(\mathbf{x})$ by

$$K(\mathbf{x}) = \left\{ i \in (I_1 \cup I_3) \setminus J(\mathbf{x}) \; : \begin{array}{c} i-1 \in J(\mathbf{x}) \\ \text{or} \\ i+1 \in J(\mathbf{x}) \end{array} \right\}$$

and let

$$K_1(\mathbf{x}) = K(\mathbf{x}) \cap I_1 \quad \text{and } K_3(\mathbf{x}) = K(\mathbf{x}) \cap I_3.$$

Note, that if $i \in K_1(\mathbf{x})$ then $i-1 \in J(\mathbf{x})$ and $i+1 \in I_2$. Since $i \notin J(\mathbf{x})$, it follows that $|x_{i+1} - x_i| > k$ and hence $|x_i - x_{i-1}| \leq k$. Similarly, if $i \in K_3(\mathbf{x})$ then $i+1 \in J(\mathbf{x})$ and $|x_{i+1} - x_i| \leq k$. Finally, for every $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$, define

$$L(\mathbf{x}) = (I_1 \cup I_3) \setminus (J(\mathbf{x}) \cup K(\mathbf{x})),$$
$$L_1(\mathbf{x}) = L(\mathbf{x}) \cap I_1, \text{ and } L_3(\mathbf{x}) = L(\mathbf{x}) \cap I_3.$$

Note, that if $i \in L_1(\mathbf{x}) \setminus \{1\}$ then $i+1 \in I_2$ and $i-1 \notin I_2 \cup J(\mathbf{x})$. Since $i \notin I_2 \cup J(\mathbf{x})$ it follows that $i-1 \notin K(\mathbf{x})$. Thus, $i-1 \in L_3(\mathbf{x})$ and $|x_i - x_{i-1}| \leq k$. Similarly, if $i \in L_3(\mathbf{x}) \setminus \{n\}$ then $i+1 \in L_1(\mathbf{x})$ and $|x_{i+1} - x_i| \leq k$.

Example 50: If $\mathbf{x} = (1, 7, 5, 3, 5, 2, 4, 9, 10, 12, 5, 4) \in \vec{\mathcal{A}}_{12,2}$ then $I_2 = \{2, 5, 8, 11\}$, $J(\mathbf{x}) = \{3, 4, 9, 12\}$, $K(\mathbf{x}) = \{10\}$, and $L(\mathbf{x}) = \{1, 6, 7\}$.

We let m be the integer $n/3$ and we define the mapping $\rho_{\mathbf{x}} : D_1 \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, \frac{3r}{4})$ as follows. For every $\mathbf{y} \in D_1$, $\mathbf{y} = (y_1, y_2, \ldots, y_m)$, let $\rho_{\mathbf{x}}(\mathbf{y}) = \mathbf{w}$, where $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is the following vector. For every $i \in [n]$, if $i \in I_2$ then

$$w_i = \begin{cases} x_i + y_{(i+1)/3}, & x_i \leq n/2, \\ x_i - y_{(i+1)/3}, & x_i > n/2. \end{cases}$$

If $i \in J(\mathbf{x})$ then

$$w_i = \begin{cases} x_i - x_{i+1} + w_{i+1}, & i \in J_1(\mathbf{x}), \\ x_i - x_{i-1} + w_{i-1}, & i \in J_3(\mathbf{x}). \end{cases}$$

If $i \in K(\mathbf{x})$ then

$$w_i = \begin{cases} x_i - x_{i-1} + w_{i-1}, & i \in K_1(\mathbf{x}), \\ x_i - x_{i+1} + w_{i+1}, & i \in K_3(\mathbf{x}). \end{cases}$$

Lastly, if $i \in L(\mathbf{x})$ then $w_i = x_i$.

Example 51: If $n = 12$, $k = 2$, and $r = 40$ then

$$D_1 = \left\{ (y_1, y_2, y_3, y_4) \; : \begin{array}{c} \sum_{i=1}^{4} y_i \leq 6, \\ \forall 1 \leq i \leq 4, \; 0 \leq y_i \leq 2 \end{array} \right\}.$$

If $\mathbf{x} = (1, 7, 5, 3, 5, 2, 4, 9, 10, 12, 5, 4) \in \vec{\mathcal{A}}_{12,2}$ and $\mathbf{y} = (1, 2, 2, 0) \in D_1$ then $\rho_{\mathbf{x}}(\mathbf{y}) = \mathbf{w}$, where $\mathbf{w}_{I_2} = (w_2, w_5, w_8, w_{11}) = (6, 7, 7, 5)$, $\mathbf{w}_{J(\mathbf{x})} = (w_3, w_4, w_9, w_{12}) = (4, 5, 8, 4)$, $\mathbf{w}_{K(\mathbf{x})} = w_{10} = 10$, and $\mathbf{w}_{L(\mathbf{x})} = (w_1, w_6, w_7) = (1, 2, 4)$. Hence, $\rho_{\mathbf{x}}(\mathbf{y}) = (1, 6, 4, 5, 7, 2, 4, 7, 8, 10, 5, 4)$.

In the next three lemmas we will prove in full details that the mapping $\rho_{\mathbf{x}}$ is well defined, i.e. we will show that $\rho_{\mathbf{x}}(\mathbf{y}) \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, \frac{3r}{4})$ for all $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$ and $\mathbf{y} \in D_1$.

Lemma 52: For any $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$ and $\mathbf{y} \in D_1$, if $\mathbf{w} = \rho_{\mathbf{x}}(\mathbf{y})$, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, then $w_i \in [n]$, for every $1 \leq i \leq n$.

Proof: We distinguish between four cases.

Case 1: $i \in I_2$. If $x_i \leq n/2$ then $w_i = x_i + y_{(i+1)/3}$, where $1 \leq x_i + y_{(i+1)/3} \leq 2n/2 - 2k \leq n$, and if $x_i > n/2$ then $w_i = x_i - y_{(i+1)/3}$, where $n/2 + 1 - (n/2 - 2k) \leq x_i - y_{(i+1)/3} \leq n$. Hence, $w_i \in [n]$.

Case 2: $i \in J(\mathbf{x})$. If $i \in J_1(\mathbf{x})$ then $i+1 \in I_2$ and $|x_{i+1} - x_i| \leq k$. If $x_{i+1} \leq n/2$ then $x_i \leq n/2 + k$ and

$$\begin{aligned} w_i &= x_i - x_{i+1} + w_{i+1} = x_i - x_{i+1} + x_{i+1} + y_{(i+2)/3} \\ &= x_i + y_{(i+2)/3}. \end{aligned}$$

It follows that $1 \leq x_i + y_{(i+2)/3} \leq n/2 + k + n/2 - 2k \leq n$. If $x_{i+1} > n/2$ then $x_i > n/2 - k$ and similarly $w_i = x_i - y_{(i+2)/3}$. It follows that $n/2 - k + 1 - (n/2 - 2k) \leq x_i - y_{(i+2)/3} \leq n$. Hence, $w_i \in [n]$. Similarly, if $i \in J_3(\mathbf{x})$ then $w_i \in [n]$.

Case 3: $i \in K(\mathbf{x})$. If $i \in K_1(\mathbf{x})$ then $i-1 \in J(\mathbf{x})$, $i-2 \in I_2$, $|x_i - x_{i-1}| \leq k$, and $|x_{i-1} - x_{i-2}| \leq k$. By the triangle inequality, it follows that $|x_i - x_{i-2}| \leq 2k$. If $x_{i-2} \leq n/2$ then $w_i = x_i + y_{(i-1)/3}$ and $x_i \leq n/2 + 2k$. It follows that $1 \leq x_i + y_{(i-1)/3} \leq n/2 + 2k + n/2 - 2k \leq n$. If $x_{i-2} > n/2$ then $w_i = x_i - y_{(i-1)/3}$ and $x_i > n/2 - 2k$. It follows that $n/2 - 2k + 1 - (n/2 - 2k) \leq x_i - y_{(i-1)/3} \leq n$. Hence, $w_i \in [n]$. Similarly, if $i \in K_3(\mathbf{x})$ then $w_i \in [n]$.

Case 4: $i \in L(\mathbf{x})$. In this case $w_i = x_i \in [n]$.

Thus, $w_i \in [n]$, for every $i \in [n]$. ■

Lemma 53: For any $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$ and $\mathbf{y} \in D_1$, if $\mathbf{w} = \rho_{\mathbf{x}}(\mathbf{y})$ then $\mathbf{w} \in \vec{\mathcal{A}}_{n,k}$.

Proof: From Lemma 52, it follows that $\mathbf{w} \in H_n$. Hence we only need to show that $|w_i - w_{i-1}| \leq k$ or $|w_{i+1} - w_i| \leq k$, for every $2 \leq i \leq n-1$. We distinguish between 4 cases.

Case 1: $i \in I_2$. In this case $i-1 \in J(\mathbf{x})$ or $i+1 \in J(\mathbf{x})$. If $i-1 \in J(\mathbf{x})$ then $w_{i-1} = x_{i-1} - x_i + w_i$ and $|x_i - x_{i-1}| \leq k$. Therefore, $|w_i - w_{i-1}| \leq k$. Similarly, if $i+1 \in J(\mathbf{x})$ then $|w_{i+1} - w_i| \leq k$.

Case 2: $i \in J(\mathbf{x})$. If $i \in J_1(\mathbf{x})$ then $i+1 \in I_2$, $w_i = x_i - x_{i+1} + w_{i+1}$, and $|x_i - x_{i+1}| \leq k$. Therefore, $|w_{i+1} - w_i| \leq k$. Similarly, if $i \in J_3(\mathbf{x})$ then $|w_i - w_{i-1}| \leq k$.

Case 3: $i \in K(\mathbf{x})$. If $i \in K_1(\mathbf{x})$ then $i-1 \in J(\mathbf{x})$, $w_i = x_i - x_{i-1} + w_{i-1}$, and $|x_i - x_{i-1}| \leq k$. Hence, $|w_i - w_{i-1}| \leq k$. Similarly, if $i \in K_3(\mathbf{x})$ then $|w_{i+1} - w_i| \leq k$.

Case 4: $i \in L(\mathbf{x})$. If $i \in L_1(\mathbf{x})$ then $i-1 \in L_3(\mathbf{x})$, $w_i = x_i$, $w_{i-1} = x_{i-1}$, and $|x_i - x_{i-1}| \leq k$. Hence, $|w_i - w_{i-1}| \leq k$.

Similarly, if $i \in L_3(\mathbf{x})$ then $|w_{i+1} - w_i| \le k$.

Thus, $|w_i - w_{i-1}| \le k$ or $|w_{i+1} - w_i| \le k$, for every $2 \le i \le n-1$. ■

Lemma 54: Let $\mathbf{y} \in D_1$. If $\mathbf{w} = \rho_{\mathbf{x}}(\mathbf{y})$, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, then $\mathbf{w} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, \frac{3r}{4})$.

Proof: By Lemma 53, it follows that $\mathbf{w} \in \vec{\mathcal{A}}_{n,k}$. It remains to show that $d_M(\mathbf{w}, \mathbf{x}) \le \frac{3r}{4}$. For $i \in L(\mathbf{x})$ we have that $w_i = x_i$ and therefore $|w_i - x_i| = 0$. For every $i \in [n] \setminus L(\mathbf{x})$ we have that $w_i \in \{x_i - y_{s(i)}, x_i + y_{s(i)}\}$, where

$$s(i) = \begin{cases} (i+1)/3, & i \in I_2, \\ (i+2)/3, & i \in J_1(\mathbf{x}), \\ i/3, & i \in J_3(\mathbf{x}), \\ (i-1)/3, & i \in K_1(\mathbf{x}), \\ (i+3)/3, & i \in K_3(\mathbf{x}). \end{cases}$$

Then,

$$d_M(\mathbf{w}, \mathbf{x}) = \sum_{i=1}^{n} |w_i - x_i| = \sum_{i \in [n] \setminus L(\mathbf{x})} |w_i - x_i|$$
$$= \sum_{i \in [n] \setminus L(\mathbf{x})} y_{s(i)} \le^{(a)} 5 \sum_{s=1}^{m} y_s \le^{(b)} 5 \frac{3r}{20} \le \frac{3r}{4},$$

where inequality (a) follows from the fact that $\sum_{i \in [n] \setminus L(\mathbf{x})} y_{s(i)}$ counts every y_s, $1 \le s \le m$, at most five times and inequality (b) follows from the definition of the set D_1.

Thus, $\mathbf{w} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, \frac{3r}{4})$. ■

We next turn to the definition of the mapping $\xi_{\mathbf{w}}$. For every $i \in I_1 \cup I_3$, let $f(i) \in [2m]$ be defined as follows.

$$f(i) = \begin{cases} i - s, & i = 3s, \text{ for some } s \in [m], \\ i - s + 1, & i = 3s - 2, \text{ for some } s \in [m]. \end{cases}$$

The mapping $\xi_{\mathbf{w}} : D_2 \to \mathcal{B}_M(\vec{\mathcal{A}}, \mathbf{w}, \frac{r}{4})$, for $\mathbf{w} \in \vec{\mathcal{A}}_{n,k}$ is defined as follows. For every $\mathbf{z} \in D_2$, $\mathbf{z} = (z_1, z_2, \ldots, z_{2m})$, let $\xi_{\mathbf{w}}(\mathbf{z}) = \mathbf{u}$, where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is the following vector. If $i \in I_2$ then $u_i = w_i$. If $i \in J(\mathbf{w})$ then

$$u_i = \begin{cases} w_i - z_{f(i)}, & w_i \ge w_\ell, w_i > z_{f(i)}, \\ z_{f(i)}, & w_i \ge w_\ell, w_i \le z_{f(i)}, \\ w_i + z_{f(i)}, & w_i < w_\ell, w_i \le n - z_{f(i)}, \\ n - z_{f(i)}, & w_i < w_\ell, w_i > n - z_{f(i)}, \end{cases}$$

where $\ell \in I_2$, $\ell = i+1$ if $i \in J_1(\mathbf{w})$ and $\ell = i-1$ if $i \in J_3(\mathbf{w})$. The motivation in this definition is to change the value of each w_i by approximately $z_{f(i)}$ such that it still remains in $[n]$ and that $|u_i - u_\ell| \le k$. The same principle will follow in the definition of u_i for $i \in K(\mathbf{w})$.

If $i \in K(\mathbf{w})$, then let $j \in J(x)$ and $\ell \in I_2$ where $j = i - 1$ and $\ell = i - 2$ if $i \in K_1(\mathbf{w})$ and $j = 1 + 1$ and $\ell = i + 2$ if $i \in K_3(\mathbf{w})$. We will define u_i according to the four possibilities of u_j. If $0 \le w_j - w_\ell \le k$ and $w_j > z_{f(j)}$, and hence $u_j = w_j - z_{f(j)}$, then

$$u_i = \begin{cases} w_i - z_{f(i)} - z_{f(j)}, & w_i \ge w_j, w_i > z_{f(i)} + z_{f(j)}, \\ z_{f(i)} + 1, & w_i \ge w_j, w_i \le z_{f(i)} + z_{f(j)}, \\ w_i + z_{f(i)}, & w_i < w_j, w_i \le n - z_{f(i)}, \\ n - z_{f(i)}, & w_i < w_j, w_i > n - z_{f(i)}. \end{cases}$$

If $0 \le w_j - w_\ell \le k$ and $w_j \le z_{f(j)}$, hence $u_j = z_{f(j)}$ and $w_i \le z_{f(j)} + k$, then

$$u_i = \begin{cases} w_i - z_{f(i)}, & w_i > z_{f(i)}, \\ z_{f(i)}, & w_i \le z_{f(i)}. \end{cases}$$

If $-k \le w_j - w_\ell < 0$ and $w_j \le n - z_{f(j)}$, hence $u_j = w_j + z_{f(j)}$, then

$$u_i = \begin{cases} w_i - z_{f(i)}, & w_i \ge w_j, w_i > z_{f(i)}, \\ z_{f(i)}, & w_i \ge w_j, w_i \le z_{f(i)}, \\ w_i + z_{f(i)} + z_{f(j)}, & w_i < w_j, w_i \le n - z_{f(i)} - z_{f(j)}, \\ n - z_{f(i)} - z_{f(j)}, & w_i < w_j, w_i > n - z_{f(i)} - z_{f(j)}. \end{cases}$$

If $-k \le w_j - w_\ell < 0$ and $w_j > n - z_{f(j)}$, hence $u_j = n - z_{f(j)}$ and $w_i \ge n - z_{f(j)} - k$, then

$$u_i = \begin{cases} w_i + z_{f(i)} + z_{f(j)}, & w_i \le n - z_{f(i)} - z_{f(j)}, \\ n - z_{f(i)} - z_{f(j)}, & w_i > n - z_{f(i)} - z_{f(j)}. \end{cases}$$

If $i \in L_1(\mathbf{w})$ and $i \ne 1$ then $i - 1 \in L_3(\mathbf{w})$. In this case, if $w_i \le n - z_{f(i)}$ then $u_i = w_i + z_{f(i)}$. If $w_{i-1} \ge w_i$, then

$$u_{i-1} = \begin{cases} w_{i-1} - z_{f(i-1)}, & w_{i-1} > z_{f(i-1)}, \\ z_{f(i-1)}, & w_{i-1} \le z_{f(i-1)}, \end{cases}$$

and if $w_{i-1} < w_i$ then

$$u_{i-1} = \begin{cases} w_{i-1} + z_{f(i-1)} + z_{f(i)}, & w_{i-1} \le n - z_{f(i-1)} - z_{f(i)}, \\ n - z_{f(i-1)} - z_{f(i)}, & w_{i-1} > n - z_{f(i-1)} - z_{f(i)}. \end{cases}$$

If $w_i > n - z_{f(i)}$ then $u_i = n - z_{f(i)}$. In this case, if $w_{i-1} \ge w_i$ then $u_{i-1} = w_{i-1} - z_{f(i-1)}$, and if $w_{i-1} < w_i$ then

$$u_{i-1} = \begin{cases} w_{i-1} + z_{f(i-1)} + z_{f(i)}, & w_{i-1} \le n - z_{f(i-1)} - z_{f(i)}, \\ n - z_{f(i-1)} - z_{f(i)}, & w_{i-1} > n - z_{f(i-1)} - z_{f(i)}. \end{cases}$$

Lastly, if $i \in L(\mathbf{w}) \cap \{1, n\}$ then

$$u_i = \begin{cases} w_i + z_{f(i)}, & w_i \le n/2, \\ w_i - z_{f(i)}, & w_i > n/2. \end{cases}$$

Example 55: If $n = 12$, $k = 2$, and $r = 40$, then

$$D_2 = \left\{ (z_1, z_2, \ldots, z_9) \ : \ \begin{array}{c} \sum_{i=1}^{9} z_i \le 5, \\ \forall 1 \le i \le 9, \ 0 \le z_i \le 1 \end{array} \right\}.$$

If $\mathbf{w} = (1, 6, 4, 5, 7, 2, 4, 7, 8, 10, 5, 4) \in \vec{\mathcal{A}}_{12,2}$ then $I_2 = \{2, 5, 8, 11\}$, $J(\mathbf{w}) = \{3, 4, 9, 12\}$, $K(\mathbf{w}) = \{10\}$, and $L(\mathbf{w}) = (1, 6, 7)$. If $\mathbf{z} = (0, 1, 1, 0, 0, 1, 0, 1, 1)$ then $\xi_{\mathbf{w}}(\mathbf{z}) = \mathbf{u}$, where $\mathbf{u}_{I_2} = (u_2, u_5, u_8, u_{11}) = (6, 7, 7, 5)$, $\mathbf{u}_{J(\mathbf{x})} = (u_3, u_4, u_9, u_{12}) = (5, 6, 7, 5)$, $\mathbf{u}_{K(\mathbf{x})} = u_{10} = 9$, and $\mathbf{u}_{L(\mathbf{x})} = (u_1, u_6, u_7) = (1, 2, 4)$. Hence, $\xi_{\mathbf{w}}(\mathbf{z}) = (1, 6, 5, 6, 7, 2, 4, 7, 7, 9, 5, 5)$.

Next, we show that the mapping $\xi_{\mathbf{w}}$ is well-defined.

Lemma 56: Let $\mathbf{z} \in D_2$. If $\xi_{\mathbf{w}}(\mathbf{z}) = \mathbf{u}$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, then $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{w}, \frac{r}{4})$.

Proof: It can be readily verified that $|u_i - u_{i-1}| \le k$ or $|u_{i+1} - u_i| \le k$, for all $2 \le i \le n-1$, and $u_i \in [n]$, for all $1 \le i \le n$. Thus, $\mathbf{u} \in \vec{\mathcal{A}}_{n,k}$. It remains to show that $d_M(\mathbf{u}, \mathbf{w}) \le \frac{r}{4}$. For every $i \in I_2$ we have that $u_i = w_i$, and therefore $|u_i - w_i| = 0$. For every $i \in J(\mathbf{w}) \cup \{1, n\}$ we have that $w_i - z_{f(i)} \le u_i \le w_i + z_{f(i)}$. For every $i \in K(\mathbf{w}) \cup L(\mathbf{w}) \setminus \{1, n\}$ we have that $w_i - z_{f(i)} - z_{g(i)} \le u_i \le w_i + z_{f(i)} + z_{g(i)}$, where

$$g(i) = \begin{cases} f(i-1), & i \in K_1(\mathbf{w}) \cup L_1(\mathbf{w}) \setminus \{1\}, \\ f(i+1), & i \in K_3(\mathbf{w}) \cup L_3(\mathbf{w}) \setminus \{n\}. \end{cases}$$

Then,

$$\begin{aligned} d_M(\mathbf{u}, \mathbf{w}) &= \sum_{i=1}^{n} |u_i - w_i| = \sum_{i \in I_1 \cup I_3} |u_i - w_i| \\ &\le \sum_{i \in I_1 \cup I_3} z_{f(i)} + \sum_{i \in K(\mathbf{w}) \cup L(\mathbf{w}) \setminus \{1,n\}} z_{g(i)} \\ &\le^{(a)} 2 \sum_{s=1}^{2m} z_s \le^{(b)} 2\frac{r}{8} \le \frac{r}{4}, \end{aligned}$$

where inequality (a) follows from the fact that each of the sums $\sum_{i \in I_1 \cup I_3} z_{f(i)}$ and $\sum_{i \in K(\mathbf{w}) \cup L(\mathbf{w}) \setminus \{1,n\}} z_{g(i)}$ counts every z_s, $1 \le s \le 2m$, at most once. Inequality (b) follows from the definition of the set D_2. Thus, $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{w}, \frac{r}{4})$. ■

Lemma 57: If $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{w}, \frac{r}{4})$ then

$$|\{\mathbf{z} \in D_2 \ : \ \xi_{\mathbf{w}}(\mathbf{z}) = \mathbf{u}\}| \le 4^{2m}.$$

Proof: We will show that there exist at most 4^{2m} possibilities to determine a vector $\mathbf{z} \in D_2$, such that $\mathbf{u} = \xi_{\mathbf{w}}(\mathbf{z})$, assuming that such a vector exists. For $i \in J(\mathbf{w}) \cup \{1, n\}$ there are at most four possibilities to determine the value of $z_{f(i)}$, given u_i and w_i. Once these elements were determined, then for every $i \in K(\mathbf{w})$, there are at most four possibilities for $z_{f(i)}$, given the vectors $\mathbf{u}, \mathbf{w}$, and the set $\{z_{f(j)} \ : \ j \in J(\mathbf{w})\}$.

For $i \in L_1(\mathbf{w}) \setminus \{1\}$, there exist at most two possibilities to determine the value of $z_{f(i)}$, given u_i and w_i. Once these elements are determined then for every $i \in L_3(\mathbf{w}) \setminus \{n\}$ there are at most two possibilities to determine the value of $z_{f(i)}$ given the vectors $\mathbf{u}, \mathbf{w}$, and the set $\{z_{f(j)} \ : \ j \in L_1(\mathbf{w}) \setminus \{1\}\}$.

Hence, for every $s \in [2m]$ there are at most four possibilities for z_s, and thus there are at most 4^{2m} possibilities for the vector $\mathbf{z}$. ■

We are now in a position to define the mapping $\psi_{\mathbf{x}} : D_1 \times D_2 \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)$. For every $\mathbf{y} \in D_1$ and $\mathbf{z} \in D_2$, let $\psi_{\mathbf{x}}(\mathbf{y}, \mathbf{z}) = \xi_{\rho_{\mathbf{x}}(\mathbf{y})}(\mathbf{z})$.

Lemma 58: If $\mathbf{y} \in D_1$ and $\mathbf{z} \in D_2$ then $\psi_{\mathbf{x}}(\mathbf{y}, \mathbf{z}) \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)$.

Proof: Let $\mathbf{w} = \rho_{\mathbf{x}}(\mathbf{y})$ and let $\mathbf{u} = \xi_{\mathbf{w}}(\mathbf{z})$. By Lemma 54 it follows that $\mathbf{w} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, \frac{3r}{4})$ and by Lemma 56 it follows that $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{w}, \frac{r}{4})$. Then $\mathbf{u} \in \vec{\mathcal{A}}_{n,k}$ and by the triangle inequality it follows that $d_M(\mathbf{u}, \mathbf{x}) \le \frac{3r}{4} + \frac{r}{4} \le r$. Thus, $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)$. ■

Proof of Lemma 35: If $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, such that there exist $\mathbf{y} \in D_1$ and $\mathbf{z} \in D_2$ for which $\mathbf{u} = \psi_{\mathbf{x}}(\mathbf{y}, \mathbf{z})$, then $\mathbf{u} = \xi_{\mathbf{w}}(\mathbf{z})$, where $\mathbf{w} = \rho_{\mathbf{x}}(\mathbf{y})$, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. For every $i \in I_2$, w_i is uniquely determined from u_i and by the definition of $\rho_{\mathbf{x}}(\mathbf{y})$, the vector $\mathbf{y}$ is uniquely determined from $\mathbf{x}$ and the set $\{w_i \ : \ i \in I_2\}$. Hence, $\mathbf{w}$ is uniquely determined from $\mathbf{u}$ and $\mathbf{x}$. By Lemma 57 it follows that there are at most 4^{2m} possibilities for the vector $\mathbf{z}$, given $\mathbf{u}$ and $\mathbf{w}$. Hence, there are at most 4^{2m} pairs $(\mathbf{y}, \mathbf{z})$ such that $\mathbf{u} = \psi_{\mathbf{x}}(\mathbf{y}, \mathbf{z})$. Thus,

$$|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)| \ge \frac{|D_1| \cdot |D_2|}{4^{\frac{2n}{3}}}.$$

■

Appendix B

The purpose of this appendix is to prove Lemma 39 from Section V. That is, for $k = \Theta(n^{\epsilon})$ and $r = \Theta(n^{\delta})$, where $0 \le \epsilon < 1$ and $1 \le \delta < 2$, there exists a constant c such that

$$\max_{\mathbf{x} \in \vec{\mathcal{A}}_{n,k}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)|\} \le c^n n^{(\delta - 1 + \epsilon)\frac{n}{2}}.$$

To this end, we first prove the following lemma.

Lemma 59: If $r = \Theta(n^{\delta})$, where $1 \le \delta$, then there exists some constant c such that for sufficiently large n

$$\binom{n+r}{r} \le (cn^{\delta-1})^n.$$

Proof: By the bounds $\left(\frac{n}{e}\right)^n \le n! \le \frac{n^{n+1}}{e^{n-1}}$, [26, p. 54], it follows that

$$\begin{aligned} \binom{n+r}{n} &= \frac{(n+r)!}{r!n!} \le \frac{n+r}{e} \cdot \frac{(n+r)^{n+r}}{r^r n^n} \\ &= \frac{n+r}{e} \cdot \left(\frac{r}{n}\right)^n \left(1 + \frac{n}{r}\right)^{n+r}. \end{aligned}$$

Hence, there exists some constant c_1 such that

$$\binom{n+r}{n} \le c_1^n n^{(\delta-1)n} \left(1 + \frac{n}{r}\right)^{n+r}.$$

There exist some constant c_2, c_3 such that

$$\left(1 + \frac{n}{r}\right)^{n+r} \le \left(\left(1 + \frac{c_2}{n^{\delta-1}}\right)^{n^{\delta-1}}\right)^{c_3 n},$$

and since

$$\left(1 + \frac{c_2}{n^{\delta-1}}\right)^{n^{\delta-1}} \le 2e^{c_2},$$

for sufficiently large n, it follows that there exists some constant c_4 such that

$$\left(1 + \frac{n}{r}\right)^{n+r} \le c_4^n.$$

Therefore, there exists some constant c such that

$$\binom{n+r}{n} \le \left(cn^{\delta-1}\right)^n.$$

■

We are now in a position to prove Lemma 39.

Proof of Lemma 39: Let $\mathbf{x} \in \vec{\mathcal{A}}_{n,k}$ and let $m = \frac{n}{2}$. For every $\mathbf{y} \in B_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)$, define the vectors $(\mathbf{u}, \mathbf{b}) \in \{0, 1, \ldots, n-1\}^m \times \{0, 1\}^m$ such that $\sum_{i=1}^{m} u_i \le r$, and $(\mathbf{z}, \mathbf{c}) \in \{0, 1, 2, \ldots, k\}^m \times \{0, 1, 2, 3\}^m$ as follows. For $1 \le i \le m$,

$$(u_i, b_i) = \begin{cases} (y_{2i-1} - x_{2i-1}, 0), & 0 \le y_{2i-1} - x_{2i-1}, \\ (x_{2i-1} - y_{2i-1}, 1), & y_{2i-1} - x_{2i-1} < 0. \end{cases}$$

For $1 \le i \le m$, if $|y_{2i} - y_{2i-1}| \le k$ then

$$(z_i, c_i) = \begin{cases} (y_{2i} - y_{2i-1}, 0), & 0 \le y_{2i} - y_{2i-1} \le k, \\ (y_{2i-1} - y_{2i}, 1), & -k \le y_{2i} - y_{2i-1} < 0. \end{cases}$$

Otherwise, if $|y_{2i} - y_{2i-1}| > k$ then

$$(z_i, c_i) = \begin{cases} (y_{2i} - y_{2i+1}, 2), & 0 \le y_{2i} - y_{2i+1} \le k, \\ (y_{2i+1} - y_{2i}, 3), & -k \le y_{2i} - y_{2i+1} < 0. \end{cases}$$

Note, that $\mathbf{y}$ is reconstructible from $(\mathbf{u}, \mathbf{b}), (\mathbf{z}, \mathbf{c})$ and $\mathbf{x}$, hence the mapping $\mathbf{y} \rightarrow ((\mathbf{u}, \mathbf{b}), (\mathbf{z}, \mathbf{c}))$ is an injection. It follows that the size of $\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)$ is at most the number of different choices of $((\mathbf{u}, \mathbf{b}), (\mathbf{z}, \mathbf{c}))$ as specified above, and therefore

$$|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)| \le 2^m \binom{m+r}{r} (4(k+1))^m.$$

By Lemma 59 it follows that there exists some constant b such that

$$\binom{m+r}{r} \le b^m m^{(\delta-1)m}.$$

Thus, there exists a constant c such that

$$|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k}, \mathbf{x}, r)| \le c^n n^{(\delta-1+\epsilon)\frac{n}{2}}.$$

■

APPENDIX C

The purpose of this appendix is to prove Lemma 41 from Section V. That is, for $\mathbf{y} \in \{0,1\}^{n-1}$, $1 \le k$, and $1 \le d \le \binom{n}{2}$, such that $d > (2k/3 + 2)n$, we have

$$E(n, k, d, \mathbf{y}) \le \frac{2^n n^2 |\vec{\mathcal{A}}_{n,k,alt}|}{\min_{\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}} \{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, r)|\}},$$

where $r = \frac{\frac{d}{2} - \left(\frac{k}{3}+1\right)n - 1}{2}$.

First, we will preset some notations and definitions. For every $\mathbf{y} \in \{0,1\}^s$ and $1 \le \ell \le s$, let $\mathbf{y}_1^\ell = (y_1, y_2, \ldots, y_\ell)$. Define $J(\mathbf{y}) \subseteq [s]$ in the following recursive manner. $J((0)) = \emptyset$, $J((1)) = \{1\}$, $J((0,0)) = \emptyset$, $J((1,0)) = \{1\}$, $J((0,1)) = \{2\}$, $J((1,1)) = \{2\}$, and for $s \ge 3$, if $y_s = 1$ then $J(\mathbf{y}) = \{s\} \cup J(\mathbf{y}_1^{s-2})$ and if $y_s = 0$ then $J(\mathbf{y}) = J(\mathbf{y}_1^{s-1})$. Notice that if $i \in J(\mathbf{y})$ then $y_i = 1$.

Example 60: Let $\mathbf{y} \in \{0,1\}^6$, where $\mathbf{y} = (1,1,0,1,1,0)$. Then $J(\mathbf{y}) = J(\mathbf{y}_1^5)$, and

$$J(\mathbf{y}_1^5) = \{5\} \cup J(\mathbf{y}_1^3) = \{5\} \cup J(\mathbf{y}_1^2) = \{2, 5\}.$$

Lemma 61: If $\mathbf{y} \in \{0,1\}^s$ does not have two consecutive zeros then

1. Either $s - 1 \in J(\mathbf{y})$ or $s \in J(\mathbf{y})$.
2. For every $2 \le i \le s-2$, if $i \notin J(\mathbf{y})$ then $i - 1 \in J(\mathbf{y})$ or $i + 1 \in J(\mathbf{y})$.

Proof: If $y_s = 1$ then $J(\mathbf{y}) = \{s\} \cup J(\mathbf{y}_1^{s-2})$ and therefore $s \in J(\mathbf{y})$ and $s - 1 \notin J(\mathbf{y})$. If $y_s = 0$ then $y_{s-1} = 1$ and $J(\mathbf{y}) = J(\mathbf{y}_1^{s-1})$. Therefore, $s \notin J(\mathbf{y})$ and $s - 1 \in J(\mathbf{y})$. This proves the first part of the lemma.

To prove the second part, we consider the two cases for the value of y_i, where $i \notin J(\mathbf{y})$. If $y_i = 1$ then we have that $y_{i+1} = 1$ and $i + 1 \in J(\mathbf{y})$. If $y_i = 0$ then since $\mathbf{y}$ does not have two consecutive zeros, $y_{i-1} = 1$ and thus $i - 1 \in J(\mathbf{y})$. ■

Define

$$I(\mathbf{y}) \stackrel{\text{def}}{=} \{i \in [1, s-2] \ : \ i \notin J(\mathbf{y}) \text{ and } i + 1 \notin J(\mathbf{y})\}.$$

Note, that if $i \in I(\mathbf{y})$ then $(y_i, y_{i+1}, y_{i+2}) = (0,1,1)$ and $i + 2 \in J(\mathbf{y})$. For a set of integers X, we denote by $1 + X$ the set $\{1 + x \ : \ x \in X\}$.

Lemma 62: Let $\mathbf{y} \in \{0,1\}^s$. If $\mathbf{y}$ does not have two consecutive zeros then

$$J(\mathbf{y}) \cup (1 + J(\mathbf{y})) \cup (1 + I(\mathbf{y})) \cup \{1, s\} = [s].$$

Proof: For $i \in [2, n-1]$, either $i \in J(\mathbf{y})$ or $i - 1 \in J(\mathbf{y})$ or $i - 1 \in I(\mathbf{y})$. Hence if $i \notin J(\mathbf{y})$ then $i = 1 + j$ for some $j \in I(\mathbf{y}) \cup J(\mathbf{y})$. ■

Recall, that for $\sigma \in A_{n,k}$ and $\mathbf{y} \in \{0,1\}^n$, $\sigma \in A_{n,k,\mathbf{y}}$ if and only if $\mu(\sigma) = \mathbf{y}$, where $(\mu(\sigma))_i = 1$ if $|\sigma(i+1) - \sigma(i)| \le k$, and $(\mu(\sigma))_i = 0$ otherwise, $1 \le i \le n-1$. This implies that $A_{n,k,\mathbf{y}} \neq \emptyset$ only if for every $1 \le i \le n-2$, $y_i = 1$ or $y_{i+1} = 1$, and hence we assume in the rest of this appendix that $\mathbf{y}$ does not have two consecutive zeros. For $\mathbf{y} \in \{0,1\}^{n-1}$, let $J(\mathbf{y}) = \{j_1, j_2, \ldots, j_{n_1}\}$, where $j_1 < j_2 < \cdots < j_{n_1}$, and let $I(\mathbf{y}) = \{i_1, i_2, \ldots, i_{n_2}\}$, where $i_1 < i_2 < \cdots < i_{n_2}$. Define the following mapping $\phi : A_{n,k,\mathbf{y}} \rightarrow \vec{\mathcal{A}}_{n,k,alt}$. For every $\sigma \in A_{n,k,\mathbf{y}}$, $\phi(\sigma) = \mathbf{x}$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, is defined as follows. For every $1 \le \ell \le n_1$, $x_{2\ell-1} = \sigma(j_\ell)$ and $x_{2\ell} = \sigma(j_\ell + 1)$. For every $1 \le \ell \le n_2$,

$$x_{2n_1+\ell} = |\sigma(i_\ell + 2) - \sigma(i_\ell + 1)|.$$

Finally, for every $2n_1 + n_2 + 1 \le \ell \le n$, $x_\ell = 1$.

Example 63: Let $\mathbf{y} = (1,1,0,1,1,0)$ and $\sigma = (4,6,7,1,3,2,5) \in A_{7,2,\mathbf{y}}$. According to the previous example, $J(\mathbf{y}) = \{2,5\}$ and hence $I(\mathbf{y}) = \{3\}$. If $\mathbf{x} = \phi(\sigma)$, where $\mathbf{x} = (x_1, x_2, \ldots, x_7)$, then $(x_1, x_2, x_3, x_4) = (\sigma(2), \sigma(3), \sigma(5), \sigma(6)) = (6,7,3,2)$, $x_5 = |\sigma(5) - \sigma(4)| = 2$, and $x_7 = x_6 = 1$. Thus, $\mathbf{x} = (6,7,3,2,2,1,1)$, which belongs to $\vec{\mathcal{A}}_{7,2,alt}$.

Lemma 64: If $\sigma \in A_{n,k,\mathbf{y}}$ then $\phi(\sigma) \in \vec{\mathcal{A}}_{n,k,alt}$.

Proof: Let $\phi(\sigma) = \mathbf{x}$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. For every $1 \le \ell \le n_1$, $j_\ell \in J(\mathbf{y})$ which implies that $y_{j_\ell} = 1$, and therefore $|\sigma(j_\ell + 1) - \sigma(j_\ell)| \le k$. Hence, $|x_{2\ell} - x_{2\ell-1}| = |\sigma(j_\ell + 1) - \sigma(j_\ell)| \le k$. For every $1 \le \ell \le n_2$, $y_{i_\ell+1} = 1$ and hence

$$x_{2n_1+\ell} = |\sigma(i_\ell + 2) - \sigma(i_\ell + 1)| \in [k].$$

Finally, for $2n_1 + n_2 + 1 \le \ell \le n$, $x_\ell = 1$. Together we conclude that for all $2n_1 + 1 \le \ell \le n$, $x_\ell \in [k]$, and thus, $\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}$. ■

Lemma 65: For every $\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}$,

$$|\{\sigma \in A_{n,k,\mathbf{y}} \ : \ \phi(\sigma) = \mathbf{x}\}| \le 2^n n^2.$$

Proof: For every $i \in [2, n-1]$, either $i - 1 \in J(\mathbf{y})$ or $i \in J(\mathbf{y})$ or $i - 1 \in I(\mathbf{y})$. If $i - 1 \in J(\mathbf{y})$ then let $1 \le \ell \le n_1$ such that $i - 1 = j_\ell$. By the definition of ϕ it follows that $\sigma(i) = x_{2\ell}$. Similarly, if $i \in J(\mathbf{y})$ then $\sigma(i) = x_{2\ell-1}$, for the unique ℓ such that $i = j_\ell$. Hence, if $i \in J(\mathbf{y})$ or $i - 1 \in J(\mathbf{y})$ then $\sigma(i)$ is uniquely determined from $\mathbf{x}$.

If $i \in [2, n-1]$ and $i, i-1 \notin J(\mathbf{y})$ then $i-1 \in I(\mathbf{y})$. From part (2) of Lemma 61 it follows that $i+1 \in J(\mathbf{y})$, and hence $\sigma(i+1)$ can be determined from $\mathbf{x}$. Let $1 \le \ell \le n_2$ such that $i-1 = i_\ell$. By the definition of ϕ it follows that $x_{2n_1+\ell} = |\sigma(i_\ell+2)-\sigma(i_\ell+1)| = |\sigma(i+1)-\sigma(i)|$. Therefore, there are at most two possibilities to determine the value of $\sigma(i)$ from $x_{2n_1+\ell}$ and $\sigma(i+1)$.

Hence, for every $i \in [2, n-1]$ there are at most two possibilities to determine $\sigma(i)$ from $\mathbf{x}$. There are at most n^2 possibilities to determine $\sigma(1)$ and $\sigma(n)$. Thus, there are at most $2^n n^2$ possibilities to determine σ. ■

Lemma 66: If $\sigma, \pi \in A_{n,k,\mathbf{y}}$ such that $d_I(\sigma,\pi) = d > (2k/3+2)n$, then $d_M(\phi(\sigma),\phi(\pi)) \ge 2r+1$, where $r = \frac{\frac{d}{2}-\left(\frac{k}{3}+1\right)n-1}{2}$.

Proof: Let $\mathbf{x} = \phi(\sigma)$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$, and $\mathbf{u} = \phi(\pi)$, $\mathbf{u} = (u_1, u_2, \dots, u_n)$. By Lemma 31 it follows that $d_M(\sigma,\pi) \ge d$. Let $J(\mathbf{y}) = \{j_1, j_2, \dots, j_{n_1}\}$ and $I(\mathbf{y}) = \{i_1, i_2, \dots, i_{n_2}\}$. By the definition of ϕ, for every $1 \le \ell \le n_1$, $x_{2\ell-1} = \sigma(j_\ell)$, $x_{2\ell} = \sigma(j_\ell+1)$, $u_{2\ell-1} = \pi(j_\ell)$, and $u_{2\ell} = \pi(j_\ell+1)$. Then

$$d_M(\mathbf{x},\mathbf{u}) \ge \sum_{i=1}^{n} |x_i - u_i| \ge \sum_{\ell=1}^{n_1} |\sigma(j_\ell)-\pi(j_\ell)| + |\sigma(j_\ell+1)-\pi(j_\ell+1)|. \tag{C.1}$$

For every $i \in [2, n-1]$ such that $i-1 \in I(\mathbf{y})$ we have that $y_i = 1$ and $i+1 \in J(\mathbf{y})$. It follows that $|\sigma(i+1)-\sigma(i)| \le k$, and $|\pi(i+1)-\pi(i)| \le k$. Therefore, by the triangle inequality

$$\begin{aligned} |\sigma(i)-\pi(i)| &\le |\sigma(i+1)-\sigma(i)| + |\pi(i+1)-\pi(i)| \\ &\quad + |\sigma(i+1)-\pi(i+1)| \\ &\le |\sigma(i+1)-\pi(i+1)| + 2k. \end{aligned} \tag{C.2}$$

Furthermore, there exists a unique $1 \le \ell \le n_1$ such that $i+1 = j_\ell$. Hence,

$$\begin{aligned} &d_M(\sigma,\pi) \\ &\le |\sigma(1)-\pi(1)| + |\sigma(n)-\pi(n)| + \sum_{i=2}^{n-1} |\sigma(i)-\pi(i)| \\ &\le^{(a)} 2n + \sum_{\ell=1}^{n_1} |\sigma(j_\ell)-\pi(j_\ell)| + |\sigma(j_\ell+1)-\pi(j_\ell+1)| \\ &\quad + \sum_{\ell=1}^{n_2} |\sigma(i_\ell+1)-\pi(i_\ell+1)| \\ &\le^{(b)} 2n + \sum_{\ell=1}^{n_1} |\sigma(j_\ell)-\pi(j_\ell)| + |\sigma(j_\ell+1)-\pi(j_\ell+1)| \\ &\quad + 2kn_2 + \sum_{\ell=1}^{n_1} |\sigma(j_\ell)-\pi(j_\ell)| \\ &\le 2n + 2kn_2 \\ &\quad + 2\sum_{\ell=1}^{n_1} |\sigma(j_\ell)-\pi(j_\ell)| + |\sigma(j_\ell+1)-\pi(j_\ell+1)| \\ &\le^{(c)} 2n(\frac{k}{3}+1) \\ &\quad + 2\sum_{\ell=1}^{n_1} |\sigma(j_\ell)-\pi(j_\ell)| + |\sigma(j_\ell+1)-\pi(j_\ell+1)|, \end{aligned}$$

where inequality (a) follows from Lemma 62 and inequality (b) follows from (C.2) and the fact that $i_\ell \in I(\mathbf{y})$ implies that $i_\ell + 2 \in J(\mathbf{y})$. Inequality (c) holds since $n_2 \le \frac{n}{3}$, which is a straightforward consequence of the inequalities $2n_1 + n_2 \le n$ and $n_2 \le n_1$. By (C.1) we conclude that

$$d_M(\sigma,\pi) \le 2n + \frac{2kn}{3} + 2d_M(\mathbf{x},\mathbf{y}),$$

and since $d_M(\sigma,\pi) \ge d$, it follows that $d_M(\mathbf{x},\mathbf{y}) \ge \frac{d}{2} - (\frac{k}{3}+1)n \ge 2r+1$. ■

Proof of Lemma 41: Let $\widetilde{d} = 2r+1$ and let $\mathcal{C} \subseteq A_{n,k,\mathbf{y}}$ be a code with minimum inversion distance d of size $E(n,k,d,\mathbf{y})$. By Lemmas 64, 65 and 66 it follows that $\phi(\mathcal{C})$ is a code in $\vec{\mathcal{A}}_{n,k,alt}$ with minimum distance at least $\widetilde{d}$ and of size at least $\frac{|\mathcal{C}|}{2^n n^2}$. By the sphere packing bound it follows that

$$|\phi(\mathcal{C})| \le \frac{|\vec{\mathcal{A}}_{n,k,alt}|}{\min_{\mathbf{x}\in\vec{\mathcal{A}}_{n,k,alt}}\{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt},\mathbf{x},r)|\}}.$$

Thus,

$$E(n,k,d,\mathbf{y}) \le \frac{2^n n^2 |\vec{\mathcal{A}}_{n,k,alt}|}{\min_{\mathbf{x}\in\vec{\mathcal{A}}_{n,k,alt}}\{|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt},\mathbf{x},r)\}}.$$

■

Appendix D

In this appendix we prove Lemma 44, that is, we will show that for every three positive integers n, k, r such that $2k < \frac{n}{2}$, and for all $\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}$

$$|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt},\mathbf{x},r)| \ge \frac{|\widetilde{D}_1(n,k,r)| \cdot |\widetilde{D}_2(n,k,r)|}{4^{\frac{n}{2}}}.$$

Recall, that $\widetilde{D}_1(n,k,r) = Q\left(\frac{n}{2}, \frac{n}{2}-k, \frac{r}{4}\right)$ and $\widetilde{D}_2(n,k,r) = Q\left(\frac{n}{2}, k, \frac{r}{2}\right)$, i.e.

$$\widetilde{D}_1(n,k,r) = \left\{ (y_1, y_2, \dots, y_{\frac{n}{2}}) \in \mathbb{Z}^{\frac{n}{2}} : \begin{array}{c} \sum_{i=1}^{\frac{n}{2}} y_i \le \frac{r}{4}, \\ 0 \le y_i \le \frac{n}{2}-k, \\ \forall 1 \le i \le \frac{n}{2}, \end{array} \right\},$$

and

$$\widetilde{D}_2(n,k,r) = \left\{ (z_1, z_2, \dots, z_{\frac{n}{2}}) \in \mathbb{Z}^{\frac{n}{2}} : \begin{array}{c} \sum_{i=1}^{\frac{n}{2}} z_i \le \frac{r}{2}, \\ 0 \le z_i \le k, \\ \forall 1 \le i \le \frac{n}{2} \end{array} \right\}.$$

We will follow similar methods to the ones used in the proof of Lemma 35. For ease of notation we denote the set $\widetilde{D}_1(n,k,r)$ by $\widetilde{D}_1$ and the set $\widetilde{D}_2(n,k,r)$ by $\widetilde{D}_2$. We will define a mapping $\widetilde{\psi}_{\mathbf{x}} : \widetilde{D}_1 \times \widetilde{D}_2 \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt},\mathbf{x},r)$ for all $\mathbf{x} \in \vec{\mathcal{A}}_{n,k,alt}$, such that

$$|\{(\mathbf{y},\mathbf{z}) \in \widetilde{D}_1 \times \widetilde{D}_2 \ : \ \widetilde{\psi}_{\mathbf{x}}(\mathbf{y},\mathbf{z}) = \mathbf{u}\}| \le 4^{\frac{n}{2}},$$

for every $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt},\mathbf{x},r)$. The mapping $\widetilde{\psi}_{\mathbf{x}}$ will be defined by two other mappings $\widetilde{\rho}_{\mathbf{x}}$ and $\widetilde{\xi}_{\mathbf{w}}$, where $\widetilde{\rho}_{\mathbf{x}} : \widetilde{D}_1 \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, \frac{r}{2})$ and $\widetilde{\xi}_{\mathbf{w}} : \widetilde{D}_2 \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{w}, \frac{r}{2})$, for $\mathbf{w} \in \vec{\mathcal{A}}_{n,k,alt}$.

Let m be the integer $n/2$. The definition of the mapping $\widetilde{\rho}_{\mathbf{x}}$ is similar to the definition of the mapping $\rho_{\mathbf{x}}$ from Appendix A. For every $\mathbf{y} \in \widetilde{D}_1$, $\mathbf{y} = (y_1, y_2, \ldots, y_m)$, let $\widetilde{\rho}_{\mathbf{x}}(\mathbf{y}) = \mathbf{w}$, where $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is defined as follows. For every $i \in [m]$,

$$w_{2i} = \begin{cases} x_{2i} + y_i, & x_{2i} \le m, \\ x_{2i} - y_i, & x_{2i} > m, \end{cases}$$

and

$$w_{2i-1} = x_{2i-1} - x_{2i} + w_{2i}.$$

Example 67: If $n = 12$, $k = 2$, and $r = 40$ then

$$\widetilde{D}_1 = \left\{ (y_1, y_2, \ldots, y_6) \ : \ \begin{array}{c} \sum_{i=1}^{6} y_i \le 10, \\ \forall 1 \le i \le 6, \ 0 \le y_i \le 4 \end{array} \right\}.$$

If $\mathbf{x} = (1, 3, 5, 3, 5, 5, 4, 6, 10, 12, 5, 4) \in \vec{\mathcal{A}}_{12,2,alt}$ and $\mathbf{y} = (4, 2, 0, 3, 1, 0) \in \widetilde{D}_1$ then we get $\widetilde{\rho}_{\mathbf{x}}(\mathbf{y}) = \mathbf{w}$, where $\mathbf{w} = (5, 7, 7, 5, 5, 5, 7, 9, 9, 11, 5, 4)$.

Lemma 68: Let $\mathbf{y} \in \widetilde{D}_1$. If $\mathbf{w} = \widetilde{\rho}_{\mathbf{x}}(\mathbf{y})$, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, then $\mathbf{w} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, \frac{r}{2})$.

Proof: It can be readily verified that $\mathbf{w} \in \vec{\mathcal{A}}_{n,k,alt}$. It remains to show that $d_M(\mathbf{w}, \mathbf{x}) \le \frac{r}{2}$. For $1 \le i \le m$ we have that $w_{2i} \in \{x_{2i} - y_i, x_{2i} + y_i\}$, and therefore $|w_{2i} - x_{2i}| = y_i$. From the definition of w_{2i-1}, we have $|w_{2i-1} - x_{2i-1}| = |w_{2i} - x_{2i}| = y_i$. Hence,

$$d_M(\mathbf{w}, \mathbf{x}) = \sum_{j=1}^{n} |w_j - x_j| = \sum_{j=1}^{n} y_{\lceil \frac{j}{2} \rceil} = 2 \sum_{i=1}^{m} y_i \le 2 \cdot \frac{r}{4} \le \frac{r}{2}.$$

Thus, $\mathbf{w} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, \frac{r}{2})$. ■

Next, we define the mapping $\widetilde{\xi}_{\mathbf{w}} : \widetilde{D}_2 \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{w}, \frac{r}{2})$, for $\mathbf{w} \in \vec{\mathcal{A}}_{n,k,alt}$. For every $\mathbf{z} \in \widetilde{D}_2$, $\mathbf{z} = (z_1, z_2, \ldots, z_m)$, let $\widetilde{\xi}_{\mathbf{w}}(\mathbf{z}) = \mathbf{u}$, where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is defined as follows. For every $i \in [m]$, $u_{2i} = w_{2i}$ and

$$u_{2i-1} = \begin{cases} w_{2i-1} - z_i, & 0 \le w_{2i-1} - w_{2i} \le k, w_{2i-1} > z_i, \\ z_i, & 0 \le w_{2i-1} - w_{2i} \le k, w_{2i-1} \le z_i, \\ w_{2i-1} + z_i, & -k \le w_{2i-1} - w_{2i} < 0, \ w_{2i-1} \le n - z_i, \\ n - z_i, & -k \le w_{2i-1} - w_{2i} < 0, \ w_{2i-1} > n - z_i. \end{cases}$$

Example 69: If $n = 12$, $k = 2$, and $r = 40$, then

$$\widetilde{D}_2 = \left\{ (z_1, z_2, \ldots, z_6) \ : \ \begin{array}{c} \sum_{i=1}^{6} z_i \le 20, \\ \forall 1 \le i \le 6, \ 0 \le z_i \le 2 \end{array} \right\}.$$

If $\mathbf{w} = (5, 7, 7, 5, 5, 5, 7, 9, 9, 11, 5, 4) \in \vec{\mathcal{A}}_{12,2,alt}$ and $\mathbf{z} = (2, 1, 2, 2, 2, 1) \in \widetilde{D}_2$ then we get $\widetilde{\xi}_{\mathbf{w}}(\mathbf{z}) = \mathbf{u}$, where $\mathbf{u} = (7, 7, 6, 5, 3, 5, 9, 9, 11, 11, 4, 4)$.

Lemma 70: Let $\mathbf{z} \in \widetilde{D}_2$. If $\mathbf{u} = \widetilde{\xi}_{\mathbf{w}}(\mathbf{z})$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, then $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{w}, \frac{r}{2})$.

Proof: From the definition of the mapping $\widetilde{\xi}_{\mathbf{w}}$ it can be readily verified that $\mathbf{u} \in \vec{\mathcal{A}}_{n,k,alt}$. It remains to show that $d_M(\mathbf{u}, \mathbf{w}) \le \frac{r}{2}$.

For every $1 \le i \le m$ we have that $u_{2i} = w_{2i}$, and therefore $|u_{2i} - w_{2i}| = 0$. For every $1 \le i \le m$ we have that

$$w_{2i-1} - z_i \le u_{2i-1} \le w_{2i-1} + z_i,$$

and therefore $|u_{2i-1} - w_{2i-1}| \le z_i$. Hence,

$$d_M(\mathbf{u}, \mathbf{w}) = \sum_{j=1}^{n} |u_j - w_j| = \sum_{i=1}^{m} |u_{2i-1} - w_{2i-1}| \le \sum_{i=1}^{m} z_i \le \frac{r}{2}.$$

Thus, $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{w}, \frac{r}{2})$. ■

Lemma 71: For every $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{w}, \frac{r}{2})$,

$$|\{\mathbf{z} \in \widetilde{D}_2 \ : \ \widetilde{\xi}_{\mathbf{w}}(\mathbf{z}) = \mathbf{u}\}| \le 4^m.$$

Proof: Let $\mathbf{u} = \widetilde{\xi}_{\mathbf{w}}(\mathbf{z})$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, for some $\mathbf{z} \in \widetilde{D}_2$, $\mathbf{z} = (z_1, z_2, \ldots, z_m)$. For every $1 \le i \le m$, there are at most four possibilities to determine z_i from u_{2i-1} and w_{2i-1}, and therefore, there are at most 4^m vectors $\mathbf{z} \in \widetilde{D}_2$ for which $\mathbf{u} = \widetilde{\xi}_{\mathbf{w}}(\mathbf{z})$. ■

We are now in a position to define the mapping $\widetilde{\psi}_{\mathbf{x}} : \widetilde{D}_1 \times \widetilde{D}_2 \to \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, r)$. For every $\mathbf{y} \in \widetilde{D}_1$ and $\mathbf{z} \in \widetilde{D}_2$, let $\widetilde{\psi}_{\mathbf{x}}(\mathbf{y}, \mathbf{z}) = \widetilde{\xi}_{\widetilde{\rho}_{\mathbf{x}}(\mathbf{y})}(\mathbf{z})$. The proof of the next lemma is similar to the proof of Lemma 58 and therefore it is omitted.

Lemma 72: If $\mathbf{y} \in \widetilde{D}_1$ and $\mathbf{z} \in \widetilde{D}_2$ then $\widetilde{\psi}_{\mathbf{x}}(\mathbf{y}, \mathbf{z}) \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, r)$.

Proof of Lemma 44: Let $\mathbf{u} \in \mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, r)$ be such that there exist $\mathbf{y} \in \widetilde{D}_1$ and $\mathbf{z} \in \widetilde{D}_2$ for which $\mathbf{u} = \widetilde{\psi}_{\mathbf{x}}(\mathbf{y}, \mathbf{z})$. Then, $\mathbf{u} = \widetilde{\xi}_{\mathbf{w}}(\mathbf{z})$, where $\mathbf{w} = \widetilde{\rho}_{\mathbf{x}}(\mathbf{y})$. For every $1 \le i \le m$, w_{2i} is uniquely determined from u_{2i} and by the definition of $\widetilde{\rho}_{\mathbf{x}}(\mathbf{y})$, the vector $\mathbf{y}$ is uniquely determined from $\mathbf{x}$ and the set $\{w_{2i} \ : \ 1 \le i \le m\}$. Hence, $\mathbf{w}$ is uniquely determined from $\mathbf{u}$ and $\mathbf{x}$. By Lemma 71 it follows that there are at most 4^m possibilities for the vector $\mathbf{z}$, given $\mathbf{u}$ and $\mathbf{w}$. Hence, there are at most 4^m pairs $(\mathbf{y}, \mathbf{z}) \in \widetilde{D}_1 \times \widetilde{D}_2$ such that $\mathbf{u} = \widetilde{\psi}_{\mathbf{x}}(\mathbf{y}, \mathbf{z})$. Thus,

$$|\mathcal{B}_M(\vec{\mathcal{A}}_{n,k,alt}, \mathbf{x}, r)| \ge \frac{|\widetilde{D}_1| \cdot |\widetilde{D}_2|}{4^m}.$$

■

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REFERENCES

[1] A. Barg and A. Mazumdar, "Codes in permutations and error correction for rank modulation," *IEEE Trans. Inf. Theory*, vol. 56, no. 7, pp. 3158–3165, Jul. 2010.

[2] A. Berman and Y. Birk, "Constrained flash memory programming," in *Proc. IEEE Symp. Inf. Theory*, St. Petersburg, Russia, Jul./Aug. 2011, pp. 2128–2132.

[3] S. Buzaglo and E. Yaakobi, "Constrained codes for rank modulation," in *Proc. IEEE Int. Symp. Inf. Theory*, Honolulu, HI, USA, Jun./Jul. 2014, pp. 2396–2400.

[4] S. Buzaglo, E. Yaakobi, T. Etzion, and J. Bruck, "Error-correcting codes for multipermutations," in *Proc. IEEE Int. Symp. Inf. Theory*, Istanbul, Turkey, Jul. 2013, pp. 724–728.

[5] P. Cappelletti and A. Modelli, *Flash Memory Reliability*. New York, NY, USA: Springer, 1999, pp. 399–441.

[6] H. D. Chadwick and L. Kurz, "Rank permutation group codes based on Kendall's correlation statistic," *IEEE Trans. Inf. Theory*, vol. 15, no. 2, pp. 306–315, Mar. 1969.

[7] T. M. Cover, "Enumerative source encoding," *IEEE Trans. Inf. Theory*, vol. 19, no. 1, pp. 73–77, Jan. 1973.

[8] P. Diaconis and R. L. Graham, "Spearman's footrule as a measure of disarray," *J. Roy. Statist. Soc. B*, vol. 39, no. 2, pp. 262–268, 1977.

[9] E. En Gad, A. Jiang, and J. Bruck, "Trade-offs between instantaneous and total capacity in multi-cell flash memories," in *Proc. IEEE Int. Symp. Inf. Theory*, Cambridge, MA, USA, Jul. 2012, pp. 990–994.

[10] E. E. Gad, E. Yaakobi, A. Jiang, and J. Bruck, "Rank-modulation rewriting codes for flash memories," in *Proc. IEEE Int. Symp. Inf. Theory*, Istanbul, Turkey, Jul. 2013, pp. 704–708.

[11] A. Jiang, R. Mateescu, M. Schwartz, and J. Bruck, "Rank modulation for flash memories," *IEEE Trans. Inf. Theory*, vol. 55, no. 6, pp. 2659–2673, Jun. 2009.

[12] A. Jiang, M. Schwartz, and J. Bruck, "Correcting charge-constrained errors in the rank-modulation scheme," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2112–2120, May 2010.

[13] M. Kendall and J. D. Gibbons, *Rank Correlation Methods*. New York, NY, USA: Oxford Univ. Press, 1990.

[14] D. E. Knuth, *The Art of Computer Programming: Sorting and Searching*, vol. 3. Reading, MA, USA: Addison-Wesley, 1998.

[15] J.-D. Lee, S.-H. Hur, and J.-D. Choi, "Effects of floating-gate interference on NAND flash memory cell operation," *IEEE Electron Device Lett.*, vol. 23, no. 5, pp. 264–266, May 2002.

[16] G. Louchard and H. Prodinger, "The number of inversions in permutations: A saddle point approach," *J. Integer Sequences*, vol. 6, no. 2, 2003, Art. ID 03.2.8.

[17] B. H. Margolius, "Permutations with inversions," *J. Integer Sequences*, vol. 4, no. 2, 2001, Art. ID 01.2.4.

[18] O. Milenkovic and B. Vasić, "Permutation (d,k) codes: Efficient enumerative coding and phrase length distribution shaping," *IEEE Trans. Inf. Theory*, vol. 46, no. 7, pp. 2671–2675, Nov. 2000.

[19] B. C. Rennie and A. J. Dobson, "On stirling numbers of the second kind," *J. Combinat. Theory*, vol. 7, no. 2, pp. 116–121, Sep. 1969.

[20] F. Sala and L. Dolecek, "Constrained permutations for rank modulation,"*IEEE Trans. Inf. Theory*, submitted for publication.

[21] F. Sala and L. Dolecek, "Constrained rank modulation schemes," in *Proc. IEEE Inf. Theory Workshop (ITW)*, Seville, Spain, Sep. 2013, pp. 1–5.

[22] F. Sala, R. Gabrys, and L. Dolecek, "Dynamic threshold schemes for multi-level non-volatile memories," *IEEE Trans. Commun.*, vol. 61, no. 7, pp. 2624–2634, Jul. 2012.

[23] D. Slepian, "Permutation modulation," *Proc. IEEE*, vol. 53, no. 3, pp. 228–236, Mar. 1965.

[24] V. Taranalli, H. Uchikawa, and P. H. Siegel, "Error analysis and intercell interference mitigation in multi-level cell flash memories," in *Proc. IEEE Int. Conf. Commun.*, Jun. 2015, pp. 1868–1873.

[25] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 2001.

[26] D. Vrajitoru and W. Knight, *Practical Analysis of Algorithms*. Cham, Switzerland: Springer International Publishing, 2014.

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