

# Nonexistence of perfect permutation codes under the Kendall $\tau$ -metric

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## Abstract

In the rank modulation scheme for flash memories, permutation codes have been studied. In this paper, we study perfect permutation codes in  $S_n$ , the set of all permutations on  $n$  elements, under the Kendall  $\tau$ -Metric. We answer one open problem proposed by Buzaglo and Etzion. That is, proving the nonexistence of perfect codes in  $S_n$ , under the Kendall  $\tau$ -metric, for more values of  $n$ . Specifically, we present the recursive formulas for the size of a ball with radius  $r$  in  $S_n$  under the Kendall  $\tau$ -metric. Further, We prove that there are no perfect  $t$ -error-correcting codes in  $S_n$  under the Kendall  $\tau$ -metric for some  $n$  and  $t = 2, 3, 4$ , or  $5$ .

**Keywords:** Flash memory, Perfect codes, Kendall  $\tau$ -Metric, Permutation codes.

## 1 Introduction

Flash memory is a non-volatile storage medium that is both electrically programmable and erasable. The rank modulation scheme for flash memories has been proposed in [2]. In this scheme, one permutation corresponds to a relative ranking of all the flash memory cells' levels. A permutation code is a nonempty subset of  $S_n$ , where  $S_n$  is the set of all the permutations over  $\{1, 2, \dots, n\}$ . Permutation codes have been studied under various metrics, such as the  $\ell_\infty$ -metric [4, 6, 7], the Ulam metric [11], and the Kendall  $\tau$ -metric [3, 5, 8, 9].

In this paper, we will focus on permutation codes under the Kendall  $\tau$ -metric. The *Kendall  $\tau$ -distance* [7] between two permutations  $\pi, \sigma \in S_n$  is the minimum number of adjacent transpositions required to obtain the permutation  $\sigma$  from  $\pi$ , where an adjacent transposition is an exchange of two distinct adjacent elements. Permutation codes under

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the Kendall  $\tau$ -distance with minimum distance  $d$  can correct up to  $\lfloor \frac{d-1}{2} \rfloor$  errors. Let  $A(n, d)$  be the size of the largest code in  $S_n$  with minimum Kendall  $\tau$ -distance  $d$ . The bounds on  $A(n, d)$  were proposed in [3, 10, 14, 15]. Some  $t$ -error-correcting codes in  $S_n$  were constructed in [1, 3, 8, 12, 13]. Buzaglo and Etzion [10] proved that there are no perfect single-error-correcting codes in  $S_n$ , where  $n > 4$  is a prime or  $4 \leq n \leq 10$ . They further [10] proposed the open problem to prove the nonexistence of perfect codes in  $S_n$ , under the Kendall  $\tau$ -metric, for more values of  $n$  and/or other distances. In this paper, we prove that there are no perfect  $t$ -error-correcting codes in  $S_n$  under the Kendall  $\tau$ -metric for some  $n$  and  $t = 2, 3, 4$ , or  $5$ . Specially, we prove that there are no perfect two-error-correcting codes in  $S_n$ , where  $n + 2 > 6$  is a prime. We also prove that there are no perfect three-error-correcting codes in  $S_n$ , where  $n + 1 > 6$  is a prime,  $n^2 + 2n - 6$  has a prime factor  $p > n$ , or  $4 \leq n \leq 33$ . We further prove that there are no perfect four-error-correcting codes in  $S_n$ , where  $n + 1 > 6$  or  $n + 2 > 7$  is a prime,  $n^2 + 3n - 12$  has a prime factor  $p > n$ , or  $5 \leq n \leq 19$ . Finally, we prove that there are no perfect five-error-correcting codes in  $S_n$ , where  $n + 7 \geq 12$  is a prime or  $n^3 + 3n^2 - 6n - 28$  has a prime factor  $p > n$ .

The rest of this paper is organized as follows. In Section 2, we will give some basic definitions for the Kendall  $\tau$ -metric and for perfect permutation codes. In Section 3, we determine the size of some balls with radius  $r$  in  $S_n$  under the Kendall  $\tau$ -metric. In Section 4, we prove the nonexistence of a perfect  $t$ -error-correcting code in  $S_n$  for some  $n$  and  $t = 2, 3, 4$ , or  $5$  by using the sphere packing upper bound. Section ?? concludes this paper.

## 2 Preliminaries

In this section we give some definitions and notations for the Kendall  $\tau$ -metric and perfect permutation codes. In addition, we summarize some important known facts.

Let  $[n]$  denote the set  $\{1, 2, \dots, n\}$ . Let  $S_n$  be the set of all the permutations over  $[n]$ . We denote by  $\pi \triangleq [\pi(1), \pi(2), \dots, \pi(n)]$  a *permutation* over  $[n]$ . For two permutations  $\sigma, \pi \in S_n$ , their multiplication  $\pi \circ \sigma$  is denoted by the composition of  $\sigma$  on  $\pi$ , i.e.,  $\pi \circ \sigma(i) = \sigma(\pi(i))$ , for all  $i \in [n]$ . Under this operation,  $S_n$  is a noncommutative *group* of size  $|S_n| = n!$ . Denote by  $\epsilon_n \triangleq [1, 2, \dots, n]$  the identity permutation of  $S_n$ . Let  $\pi^{-1}$  be the *inverse* element of  $\pi$ , for any  $\pi \in S_n$ . For an unordered pair of distinct numbers  $i, j \in [n]$ , this pair forms an inversion in a permutation  $\pi$  if  $i < j$  and simultaneously  $\pi(i) > \pi(j)$ .

Given a permutation  $\pi = [\pi(1), \pi(2), \dots, \pi(i), \pi(i+1), \dots, \pi(n)] \in S_n$ , an adjacent transposition is an exchange of two adjacent elements  $\pi(i), \pi(i+1)$ , resulting in the permutation  $[\pi(1), \pi(2), \dots, \pi(i+1), \pi(i), \dots, \pi(n)]$  for some  $1 \leq i \leq n-1$ . For any two permutations  $\sigma, \pi \in S_n$ , the Kendall  $\tau$ -distance between two permutations  $\pi, \sigma$ , denoted by  $d_K(\pi, \sigma)$ , is the minimum number of adjacent transpositions required to obtain the permutation  $\sigma$  from  $\pi$ . The expression for  $d_K(\pi, \sigma)$  [3] is as follows:

$$d_K(\sigma, \pi) = |\{(i, j) : \sigma^{-1}(i) < \sigma^{-1}(j) \wedge \pi^{-1}(i) > \pi^{-1}(j)\}|.$$

For  $\pi \in S_n$ , the Kendall  $\tau$ -weight of  $\pi$ , denoted by  $w_K(\pi)$ , is defined as the Kendall

$\tau$ -distance between  $\pi$  and the identity permutation  $\epsilon_n$ . Clearly,  $w_K(\pi)$  is the number of inversions in the permutation  $\pi$ .

**Definition 1.** For  $1 \leq d \leq \binom{n}{2}$ ,  $C \subset S_n$  is an  $(n, d)$ -permutation code under the Kendall  $\tau$ -metric, if  $d_K(\sigma, \pi) \geq d$  for any two distinct permutations  $\pi, \sigma \in C$ .

For a permutation  $\pi \in S_n$ , the Kendall  $\tau$ -ball of radius  $r$  centered at  $\pi$ , denoted as  $B_K^n(\pi, r)$ , is defined by  $B_K^n(\pi, r) \triangleq \{\sigma \in S_n | d_K(\sigma, \pi) \leq r\}$ . For a permutation  $\pi \in S_n$ , the Kendall  $\tau$ -sphere of radius  $r$  centered at  $\pi$ , denoted as  $S_K^n(\pi, r)$ , is defined by  $S_K^n(\pi, r) \triangleq \{\sigma \in S_n | d_K(\sigma, \pi) = r\}$ . The size of a Kendall  $\tau$ -ball or a  $\tau$ -sphere of radius  $r$  does not depend on the center of the ball under the Kendall  $\tau$ -metric. Thus, we denote the size of  $B_K^n(\pi, r)$  and  $S_K^n(\pi, r)$  as  $B_K^n(r)$  and  $S_K^n(r)$ , respectively. We denote the largest size of an  $(n, d)$ -permutation code under the Kendall  $\tau$ -metric as  $A_K(n, d)$ . The sphere-packing bound for permutation codes under the Kendall  $\tau$ -metric are as follows:

**Proposition 1.** [3, Theorems 17 and 18]

$$A_K(n, d) \leq \frac{n!}{B_K^n(\lfloor \frac{d-1}{2} \rfloor)}.$$

When  $d = 2r + 1$ , an  $(n, 2r + 1)$ -permutation code  $C$  under the Kendall  $\tau$ -metric is called a perfect permutation code under the Kendall  $\tau$ -metric if it attains the sphere-packing bound, i.e.,  $|C| \cdot B_K^n(r) = n!$ . That is, the balls with radius  $r$  centered at the codewords of  $C$  form a partition of  $S_n$ . A perfect  $(n, 2r + 1)$ -permutation code under the Kendall  $\tau$ -metric is also called a perfect  $r$ -error-correcting code under the Kendall  $\tau$ -metric.

In [10], Buzaglo and Etzion proved that there does not exist a perfect one-error-correcting code under the Kendall  $\tau$ -metric if  $n > 4$  is a prime or  $4 \leq n \leq 10$ . Based on the above definitions and notations, we will prove the nonexistence of a perfect  $t$ -error-correcting code in  $S_n$  under the Kendall  $\tau$ -metric for some  $n$  and  $t = 2, 3, 4$ , or  $5$  by using the sphere-packing upper bound in the following sections.

### 3 The size of a ball or a sphere with radius $r$ in $S_n$ under the Kendall $\tau$ -metric

In this section, we compute the size of a ball or a sphere with radius  $r$  in  $S_n$  under the Kendall  $\tau$ -metric and give recursive formulas of  $B_K^n(r)$  and  $S_K^n(r)$ , respectively. Since  $B_K^n(r)$  does not depend on the center of the ball, we consider the ball  $B_K^n(\epsilon_n, r)$  which is a ball with radius  $r$  centered at the identity permutation  $\epsilon_n$  and denote by  $S_K^n(\epsilon_n, r) \triangleq \{\sigma \in S_n | d_K(\sigma, \epsilon_n) = w_k(\sigma) = r\}$  the sphere centered at  $\epsilon_n$  and of radius  $r$ .

#### 3.1 The size of a sphere of radius $r$ in $S_n$ under the Kendall $\tau$ -metric

In order to give the property of  $S_K^n(r)$ , we require some notations and lemmas in [10]. For a permutation  $\pi = [\pi(1), \pi(2), \dots, \pi(n)] \in S_n$ , the *reverse* of  $\pi$  is the permutation

$\pi^r \triangleq [\pi(n), \pi(n-1), \dots, \pi(2), \pi(1)]$ . For all  $\pi \in S_n$ , we have  $w_K(\pi) \leq \binom{n}{2}$ . For convenience, we denote  $S_K^n(r) = 0$  for  $r \geq \binom{n}{2} + 1$ .

**Lemma 1.** [10, Lemma 1] For every  $\pi, \epsilon_n \in S_n$ ,

$$d_K(\epsilon_n, \pi) + d_K(\epsilon_n, \pi^r) = w_K(\pi) + w_K(\pi^r) = d_K(\pi, \pi^r) = \binom{n}{2}. \quad (1)$$

By Lemma 1, we can obtain the following lemma.

**Lemma 2.** For any  $0 \leq i \leq \lfloor \frac{\binom{n}{2}}{2} \rfloor$ ,

$$S_K^n(i) = S_K^n\left(\binom{n}{2} - i\right). \quad (2)$$

*Proof.* Let  $m = \binom{n}{2}$ . We just need to prove that  $|S_K^n(\epsilon_n, i)| = |S_K^n(\epsilon_n, m - i)|$ . First we define a function  $f : S_K^n(\epsilon_n, i) \rightarrow S_K^n(\epsilon_n, m - i)$ , where  $f(\pi) = \pi^r$  for any  $\pi \in S_K^n(\epsilon_n, i)$ .

If  $\pi \in S_K^n(\epsilon_n, i)$ , then  $w_K(\pi) = i$ . By (1),  $w_K(\pi^r) = \binom{n}{2} - i = m - i$ . Hence,  $f(\pi) \in S_K^n(\epsilon_n, m - i)$ . Moreover, we can easily prove that the function  $f$  is reasonable and bijection. Thus,  $S_K^n(i) = S_K^n(\binom{n}{2} - i)$ .  $\square$

When  $i = 0$  or  $1$ ,  $S_K^n(0) = 1$  and  $S_K^n(1) = n - 1$ . We will further give a recursive formula of  $S_K^n(r)$  in the following lemma.

**Lemma 3.** For all  $4 \leq n$  and  $2 \leq i \leq n - 1$ ,

$$S_K^n(i) = \sum_{j=0}^i S_K^{n-1}(j). \quad (3)$$

Moreover, for all  $5 \leq n$  and  $n \leq i \leq \lfloor \frac{\binom{n}{2}}{2} \rfloor$ ,

$$S_K^n(i) = \sum_{j=i-(n-1)}^i S_K^{n-1}(j). \quad (4)$$

*Proof.* When  $4 \leq n$  and  $2 \leq i \leq n - 1$ , we define  $S_K^n(\epsilon_n, i, j) \triangleq \{\pi \in S_K^n(\epsilon_n, i) | \pi(j) = n\}$  for  $n - i \leq j \leq n$ , i.e.,  $\pi \in S_K^n(\epsilon_n, i)$  is an element of  $S_K^n(\epsilon_n, i, j)$  if  $n$  appears at the  $j$ th position of  $\pi$ . For  $\pi \in S_K^n(\epsilon_n, i)$ , the number of inversions in the permutation  $\pi$  is  $i$ . If  $\pi(j) = n$ ,  $(\pi(k), n)$  is an inversion for all  $j + 1 \leq k \leq n$ . Hence, for any  $\pi \in S_K^n(\epsilon_n, i)$ ,  $n$  can only appear at the  $j$ th position of  $\pi$  for every  $n - i \leq j \leq n$ . So, we obtain that  $S_K^n(\epsilon_n, i) = \cup_{j=n-i}^n S_K^n(\epsilon_n, i, j)$ .

For all  $n - i \leq j \leq n$ , we define  $f_j : S_K^n(\epsilon_n, i, j) \rightarrow S_K^{n-1}(\epsilon_{n-1}, i - (n - j))$ , where  $f_j(\pi) = [\pi(1), \pi(2), \dots, \pi(j - 1), \pi(j + 1), \dots, \pi(n)]$  for any  $\pi \in S_K^n(\epsilon_n, i, j)$ . That is, we delete the element  $n$  of  $\pi$  to obtain  $f_j(\pi)$ . Obviously,  $f_j$  is injective. For  $\pi_1 \in S_K^{n-1}(\epsilon_{n-1}, i - (n - j))$ ,

we define  $\pi$  such that  $\pi(k) = \pi_1(k)$  for  $1 \leq k \leq j-1$ ,  $\pi(j) = n$ , and  $\pi(k) = \pi_1(k-1)$  for  $j+1 \leq k \leq n$ . Then,  $\pi \in S_n$  and  $w_K(\pi) = w_K(\pi_1) + (n-j) = i$ . Thus,  $\pi \in S_K^n(\epsilon_n, i, j)$  and  $f_j(\pi) = \pi_1$ . So, we obtain that  $f_j$  is bijection for all  $n-i \leq j \leq n$ .

Since all the set  $S_K^n(\epsilon_n, i, j)$  are pairwise disjoint and all the  $f_j$  are bijection for all  $n-i \leq j \leq n$ , we have

$$\begin{aligned} S_K^n(i) &= |S_K^n(\epsilon_n, i)| = |\cup_{j=n-i}^n S_K^n(\epsilon_n, i, j)| = \sum_{j=n-i}^n |S_K^n(\epsilon_n, i, j)| \\ &= \sum_{j=n-i}^n |S_K^{n-1}(\epsilon_{n-1}, i - (n-j))| = \sum_{j=0}^i S_K^{n-1}(j). \end{aligned}$$

Similarly, for all  $5 \leq n$  and  $n \leq i \leq \lfloor \frac{\binom{n}{2}}{2} \rfloor$ , then  $\lfloor \frac{\binom{n}{2}}{2} \rfloor \leq \binom{n-1}{2}$ . Thus, for all  $i - (n-1) \leq j \leq \lfloor \frac{\binom{n}{2}}{2} \rfloor$ ,  $S_K^{n-1}(j)$  exists. So, we also prove that

$$S_K^n(i) = \sum_{j=i-(n-1)}^i S_K^{n-1}(j).$$

□

Furthermore, we give the recursive formula of  $S_K^n(i)$  for all  $4 \leq n$  and  $4 \leq i \leq n-1$  in the following lemma. For convenience, for any function  $f(t)$  and two positive integers  $i < t$ , we denote  $\sum_{l=t}^i f(l) = 0$ .

**Lemma 4.** *For all  $4 \leq n$  and  $4 \leq i \leq n-1$ , there exists a unique integer  $t$  such that  $\binom{t-1}{2} < i \leq \binom{t}{2}$  and  $t \geq 4$ . Then, we have*

$$S_K^n(i) = S_K^t\left(\binom{t}{2} - i\right) + \sum_{l=t}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j) + \sum_{l=i}^{n-1} \sum_{j=0}^{i-1} S_K^l(j). \quad (5)$$

*Proof.* When  $4 \leq n$  and  $4 \leq i \leq n-1$ , by (3), we have

$$S_K^n(i) - S_K^{n-1}(i) = \sum_{j=0}^{i-1} S_K^{n-1}(j). \quad (6)$$

In (6), we set  $n$  to  $i+1, \dots, n$  and obtain  $n-i$  equations, respectively. Then by summing all the equations, we have

$$S_K^n(i) - S_K^i(i) = \sum_{l=i}^{n-1} \sum_{j=0}^{i-1} S_K^l(j). \quad (7)$$

For  $j < i$  and  $i < n$ , if  $S_K^n(j)$  and  $S_K^i(i)$  are known, then by (7) we can compute  $S_K^n(i)$ . In the following, we will compute  $S_K^i(i)$ . By (4), for  $i \leq \binom{i-1}{2}$  (i.e.,  $4 \leq i$ ), we obtain that

$$S_K^i(i) - S_K^{i-1}(i) = \sum_{j=1}^{i-1} S_K^{i-1}(j). \quad (8)$$

For  $4 \leq i$ , we can find an integer  $t$  such that  $\binom{t-1}{2} < i \leq \binom{t}{2}$  and  $t \geq 4$ . Then,  $\binom{t}{2} + \frac{(t-1)(t-4)}{2} < 2i$  and  $t < \binom{t-1}{2}$  for  $5 \leq t$ . When  $i = 4$ , we have  $t = 4$ . When  $5 \leq i$ , we have  $4 \leq t$ ,  $i \leq \binom{t}{2} < 2i$ , and  $t < i$ .

Thus, we obtain

$$0 \leq \binom{t}{2} - i < i. \quad (9)$$

When  $i = 4$ ,  $S_K^4(4) = S_K^4(\binom{4}{2}) - 4 = S_K^4(2)$ .

Similarly, when  $4 < i$ , in (4), we set  $n$  to  $t + 1, \dots, i$  and obtain  $i - t$  equations, respectively. By summing all the equations, we have

$$S_K^i(i) - S_K^t(i) = \sum_{l=t}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j). \quad (10)$$

Combining (2), (9), and (10), we have

$$S_K^i(i) = S_K^t\left(\binom{t}{2} - i\right) + \sum_{l=t}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j). \quad (11)$$

When  $4 \leq i$ , we also have  $S_K^i(i) = S_K^t\left(\binom{t}{2} - i\right) + \sum_{l=t}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j)$ . When  $i = t = 4$ , the second term (i.e.,  $\sum_{l=4}^3 \sum_{j=i-l}^{i-1} S_K^l(j)$ ) is zero. Finally, by (7) and (11), we can obtain the expression of  $S_K^n(i)$  in the above lemma.  $\square$

Specifically, we give the formulas of  $S_K^n(2)$  and  $S_K^n(3)$  for all  $3 \leq n$  as follows.

**Lemma 5.** *For all  $3 \leq n$ , we have*

$$S_K^n(2) = \frac{n(n-1)}{2} - 1, \quad (12)$$

$$S_K^n(3) = \frac{n^3 - 7n}{6}. \quad (13)$$

*Proof.* When  $i = 2$ , by (6), we have

$$S_K^n(2) - S_K^2(2) = \sum_{l=2}^{n-1} \sum_{j=0}^1 S_K^l(j). \quad (14)$$

Since  $S_K^n(0) = 1$ ,  $S_K^n(1) = n - 1$  and  $S_K^2(2) = 0$ , by (14), we have

$$S_K^n(2) = \sum_{l=2}^{n-1} \sum_{j=0}^1 S_K^l(j) = \sum_{l=2}^{n-1} l = \frac{n(n-1)}{2} - 1. \quad (15)$$

Similarly, when  $i = 3$ , by (6), we have

$$S_K^n(3) - S_K^3(3) = \sum_{l=3}^{n-1} \sum_{j=0}^2 S_K^l(j). \quad (16)$$

Since  $S_K^n(0) = 1$ ,  $S_K^n(1) = n - 1$ ,  $S_K^n(2) = \frac{n(n-1)}{2} - 1$ , and  $S_K^3(3) = 1$ , by (16), we have

$$S_K^n(3) = S_K^3(3) + \sum_{l=3}^{n-1} \sum_{j=0}^2 S_K^l(j) = 1 + \sum_{l=3}^{n-1} \frac{l^2 + l - 2}{2} = \frac{n^3 - 7n}{6}. \quad (17)$$

According to (15) and (17), we can obtain the expressions of  $S_K^n(2)$  and  $S_K^n(3)$  as (12) and (13), respectively.  $\square$

Here, we easily obtain  $S_K^2(0) = S_K^2(1) = 1$ . By Lemma 5, when  $n = 3$ , we have  $S_K^3(0) = 1$ ,  $S_K^3(1) = 2$ ,  $S_K^3(2) = 2$ , and  $S_K^3(3) = 1$ . By Lemma 5 and Lemma 2, we have  $S_K^4(0) = 1$ ,  $S_K^4(1) = 3$ ,  $S_K^4(2) = 5$ ,  $S_K^4(3) = 6$ ,  $S_K^4(4) = 5$ ,  $S_K^4(5) = 3$ , and  $S_K^4(6) = 1$ .

If all the  $S_K^n(j)$  for all  $n$  and  $j \leq i - 1$  are known, by Lemma 4, we can compute  $S_K^n(i)$  for  $4 \leq n$  and  $4 \leq i \leq n - 1$ . Next we present an example to compute  $S_K^n(i)$  in Lemma 4.

**Example 1.** When  $i = 4$ ,  $\binom{3}{2} < 4 \leq \binom{4}{2}$ . Then, we obtain  $t = 4$  in Lemma 4. Furthermore, by (5), we have

$$S_K^n(4) = S_K^4\left(\binom{4}{2} - 4\right) + \sum_{l=4}^3 \sum_{j=i-l}^{i-1} S_K^l(j) + \sum_{l=4}^{n-1} \sum_{j=0}^3 S_K^l(j).$$

By Lemma 5, we have  $S_K^4\left(\binom{4}{2} - 4\right) = S_K^4(2) = 5$ . Thus,

$$S_K^n(4) = 5 + \sum_{l=4}^{n-1} \left(1 + (l-1) + \frac{l(l-1)}{2} - 1 + \frac{l^3 - 7l}{6}\right) = \frac{n(n+1)(n^2 + n - 14)}{24}. \quad (18)$$

In the following, we give the recursive formula of  $S_K^n(i)$  for all  $5 \leq n$  and  $n \leq i \leq \lfloor \frac{n}{2} \rfloor$ .

**Lemma 6.** For all  $5 \leq n$  and  $n \leq i \leq \lfloor \frac{n}{2} \rfloor$ , there exists a unique integer  $t$  such that  $\binom{t-1}{2} < i \leq \binom{t}{2}$  and  $t \geq 4$ . Then, we have

$$S_K^n(i) = S_K^t\left(\binom{t}{2} - i\right) + \sum_{l=t}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j) - \sum_{l=n}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j). \quad (19)$$

*Proof.* When  $5 \leq n$  and  $n \leq i \leq \lfloor \frac{n}{2} \rfloor$ , in (4), we set  $n$  to  $n + 1, \dots, i$ , respectively. Then we obtain  $n - i$  equations and sum all the equations. Thus, we have

$$S_K^i(i) - S_K^n(i) = \sum_{l=n}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j). \quad (20)$$

By (11) and (20), we have

$$S_K^n(i) = S_K^t\left(\binom{t}{2} - i\right) + \sum_{l=t}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j) - \sum_{l=n}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j).$$

When  $i = n$ , the third term (i.e.,  $\sum_{l=n}^{i-1} \sum_{j=i-l}^{i-1} S_K^l(j)$ ) is zero.  $\square$

**Example 2.** When  $i = 5$  and  $n = 5$ , we have  $\binom{3}{2} < 5 \leq \binom{4}{2}$ . Then, we obtain  $t = 4$  in Lemma 6. Furthermore, by (19), we have

$$S_K^5(5) = S_K^4\left(\binom{4}{2} - 5\right) + \sum_{l=4}^4 \sum_{j=5-l}^4 S_K^l(j).$$

Thus,  $S_K^5(5) = S_K^4(1) + \sum_{j=1}^4 S_K^4(j) = 3 + (3 + 5 + 6 + 5) = 22$ .

For every  $6 \leq n$ , due to  $i = 5 \leq n - 1$ ,  $S_K^n(5)$  can be computed by Lemma 4.

Hence, if  $S_K^n(j)$  are known for all  $1 \leq j \leq i - 1$  and  $n$ , we will compute  $S_K^n(i)$  for all  $n$  in the next two steps. For  $5 \leq i$ , there exists a unique integer  $t$  such that  $\binom{t-1}{2} < i \leq \binom{t}{2}$ . Then, for every  $2 \leq l \leq t - 1$ ,  $S_K^l(i) = 0$ . First, when  $t \leq l \leq i$ , if  $i > \lfloor \frac{\binom{l}{2}}{2} \rfloor$ , we have  $S_K^l(i) = S_K^l(\binom{l}{2} - i)$  where  $\binom{l}{2} - i < i$ ; otherwise, by Lemma 6, we compute  $S_K^l(i)$  for  $i \leq \lfloor \frac{\binom{l}{2}}{2} \rfloor$ . Second, when  $i + 1 \leq l$ , we compute  $S_K^l(i)$  by Lemma 4.

When  $i = 5$ , we can compute  $S_K^n(5)$  for all  $n$ . Here,  $t = 4$ . Then,  $S_K^4(5) = S_K^4(\binom{4}{2} - 5) = S_K^4(1) = 3$  and  $S_K^5(5) = 22$  by Lemma 6 in Example 2. In the following, we will give the formula of  $S_K^n(5)$  for all  $6 \leq n$  by Lemma 4.

**Example 3.** When  $i = 5$  and  $6 \leq n$ , by Lemma 4 and (7), we have

$$S_K^n(5) = S_K^5(5) + \sum_{l=5}^{n-1} \sum_{j=0}^4 S_K^l(j).$$

By Examples 1 and 2 and Lemma 5, we have

$$\begin{aligned} S_K^n(5) &= 22 + \sum_{l=5}^{n-1} \left(1 + (l-1) + \frac{l(l-1)}{2} - 1 + \frac{l^3 - 7l}{6} + \frac{l(l+1)(l^2 + l - 14)}{24}\right) \\ &= \frac{(n-1)(n^4 + 6n^3 - 9n^2 - 74n - 120)}{120} \end{aligned} \quad (21)$$

for all  $5 \leq n$ .

By Lemmas 2, 4, and 6, we can obtain the property of  $S_K^n(i)$  for all  $6 \leq i$  and  $n$  as follows.

**Proposition 2.** When  $6 \leq i$ , we can compute  $S_K^n(i)$  for all  $5 \leq n$  by using Lemmas 2, 4, and 6.

*Proof.* For all  $0 \leq i \leq 5$  and  $3 \leq n$ , all the  $S_K^n(i)$  are computed. We can compute  $S_K^n(i)$  for all  $n$  by using  $S_K^n(j)$  for all  $1 \leq j \leq i - 1$  and  $n$ .

First, we find an integer  $t$  such that  $\binom{t-1}{2} < i \leq \binom{t}{2}$ . For every  $t \leq l \leq i$ , if  $i > \lfloor \frac{\binom{l}{2}}{2} \rfloor$ , we have  $S_K^l(i) = S_K^l(\binom{l}{2} - i)$  where  $\binom{l}{2} - i < i$ ; else if  $i \leq \lfloor \frac{\binom{l}{2}}{2} \rfloor$ , we compute  $S_K^l(i)$  by Lemma 6. Second, for every  $i + 1 \leq l$ , we compute  $S_K^l(i)$  by Lemma 4. So, we can obtain  $S_K^n(i)$  for all  $5 \leq n$  and  $6 \leq i$ .  $\square$



### 3.2 The size of a ball of radius $r$ in $S_n$ under the Kendall $\tau$ -metric

In this subsection, we will give the size of a ball with radius  $r$  in  $S_n$  under the Kendall  $\tau$ -metric and give recursive formula of  $B_K^n(r)$  by using  $S_K^n(r)$ . We easily obtain the following lemma about the relationship between  $B_K^n(r)$  and  $S_K^n(r)$ .

**Lemma 7.** *For any  $0 \leq r \leq \binom{n}{2}$ , we have*

$$B_K^n(r) = \sum_{l=0}^r S_K^n(l). \quad (22)$$

Given  $S_K^n(i)$  for all  $0 \leq i \leq r-1$ , by Lemmas 4, 6 and 7, we easily obtain the recursion formula of  $B_K^n(r)$  in the following theorem.

**Theorem 1.** *Suppose  $S_K^n(i)$  are known for all  $0 \leq i \leq r-1$  and  $5 \leq n$ . If  $4 \leq r \leq \lfloor \frac{\binom{n}{2}}{2} \rfloor$ , there exists a unique integer  $t$  such that  $\binom{t-1}{2} < r \leq \binom{t}{2}$ . When  $4 \leq r \leq n-1$ , we have*

$$B_K^n(r) = \sum_{l=0}^{r-1} S_K^n(l) + S_K^t\left(\binom{t}{2} - r\right) + \sum_{l=t}^{r-1} \sum_{j=r-l}^{r-1} S_K^l(j) + \sum_{l=r}^{n-1} \sum_{j=0}^{r-1} S_K^l(j). \quad (23)$$

When  $n \leq r \leq \lfloor \frac{\binom{n}{2}}{2} \rfloor$ , we have

$$B_K^n(r) = \sum_{l=0}^{r-1} S_K^n(l) + S_K^t\left(\binom{t}{2} - r\right) + \sum_{l=t}^{r-1} \sum_{j=r-l}^{r-1} S_K^l(j) - \sum_{l=n}^{r-1} \sum_{j=r-l}^{r-1} S_K^l(j). \quad (24)$$

Specially, we have  $B_K^n(0) = 1$  and  $B_K^n(1) = n$ . When  $r = 2$ , for all  $n \geq 2$ , we have

$$B_K^n(2) = \sum_{l=0}^2 S_K^n(l) = (1 + n - 1 + \frac{n(n-1)}{2} - 1) = \frac{(n+2)(n-1)}{2}. \quad (25)$$

When  $r = 3$ , for all  $n \geq 3$ , we have

$$B_K^n(3) = \sum_{l=0}^3 S_K^n(l) = (1 + n - 1 + \frac{n(n-1)}{2} - 1 + \frac{n^3 - 7n}{6}) = \frac{(n+1)(n^2 + 2n - 6)}{6}. \quad (26)$$

**Example 4.** *When  $r = 4$  and  $4 \leq n$ , by Example 1 and Theorem 1, we have*

$$\begin{aligned} B_K^n(4) &= \sum_{l=0}^3 S_K^n(l) + S_K^4\left(\binom{4}{2} - 4\right) + \sum_{l=4}^3 \sum_{j=4-l}^{4-1} S_K^l(j) + \sum_{l=4}^{n-1} \sum_{j=0}^3 S_K^l(j) \\ &= \frac{(n+2)(n+1)(n^2 + 3n - 12)}{24}. \end{aligned} \quad (27)$$

Moreover, when  $r = 5$  and  $5 \leq n$ , by Example 3 and Theorem 1, we have

$$\begin{aligned} B_K^n(5) &= \sum_{l=0}^4 S_K^n(l) + S_K^n(5) \\ &= \frac{(n+7)n(n^3 + 3n^2 - 6n - 28)}{120}. \end{aligned} \quad (28)$$

When  $r \geq 6$ , we can compute  $B_K^n(r)$  by using Proposition 2 and Theorem 1.

## 4 The nonexistence of a perfect $t$ -error-correcting code in $S_n$ under the Kendall $\tau$ -metric for some $n$ and $t = 2, 3, 4$ , or $5$

In this section, we will prove the nonexistence of a perfect  $t$ -error-correcting code in  $S_n$  under the Kendall  $\tau$ -metric for some  $n$  and  $t = 2, 3, 4$ , or  $5$  by using the sphere-packing upper bound. By Proposition 1, we give the necessary condition of the existence of a perfect  $t$ -error-correcting code in  $S_n$  under the Kendall  $\tau$ -metric.

**Lemma 8.** *For any  $0 \leq t \leq \binom{n}{2}$ , if there exists one perfect  $t$ -error-correcting code  $C$  in  $S_n$  under the Kendall  $\tau$ -metric. Then, we must have*

$$B_K^n(t) \cdot |C| = n!. \quad (29)$$

*That is, the necessary condition of the existence of a perfect  $t$ -error-correcting code in  $S_n$  under the Kendall  $\tau$ -metric is  $B_K^n(t)|n|$ .*

*Proof.* By the sphere-packing upper bound in Proposition 1, if there exists one perfect  $t$ -error-correcting code  $C$  in  $S_n$  under the Kendall  $\tau$ -metric, we must have  $B_K^n(t) \cdot |C| = n!$ . Thus,  $B_K^n(t)|n|$ . So, the necessary condition of the existence of a perfect  $t$ -error-correcting code in  $S_n$  under the Kendall  $\tau$ -metric is  $B_K^n(t)|n|$ .  $\square$

According to Lemma 8, we have the following theorem which illustrate the nonexistence of a perfect  $t$ -error-correcting code in  $S_n$  under the Kendall  $\tau$ -metric.

**Theorem 2.** *For any  $0 \leq t \leq \binom{n}{2}$ , if  $B_K^n(t)$  has a prime factor  $p > n$ , then there does not exist one perfect  $t$ -error-correcting code in  $S_n$  under the Kendall  $\tau$ -metric.*

*Proof.* By Lemma 8, the necessary condition of the existence of a perfect  $t$ -error-correcting code in  $S_n$  under the Kendall  $\tau$ -metric is  $B_K^n(t)|n|$ . Since  $B_K^n(t)$  has a prime factor  $p > n$ , we have  $B_K^n(t) \nmid n!$ . So, we prove the above result.  $\square$

In the following, we will discuss the nonexistence of a perfect  $t$ -error-correcting code in  $S_n$  for some  $n$  and  $t = 2, 3, 4$ , or  $5$  by using Theorem 2.

When  $t = 2$ , by (25), we have  $B_K^n(2) = \frac{(n+2)(n-1)}{2}$ . By Theorem 2, we can prove the nonexistence of a perfect two-error-correcting code in  $S_n$ , where  $n+2 > 6$  is a prime.

When  $t = 3$ , by (26), we have  $B_K^n(3) = \frac{(n+1)(n^2+2n-6)}{6}$ . First, if  $n + 1 > 6$  is a prime, then  $B_K^n(3)$  have a prime factor  $n + 1 > n$ . Second, we compute  $n^2 + 2n - 6$  for  $4 \leq n \leq 33$  and obtain that  $(n + 1)(n^2 + 2n - 6)$  has a prime factor  $p > n$  except  $n = 13$  and  $n = 26$ . If  $n = 13$ ,  $B_K^{13}(3) = 441 = 9 \times 7^2$ . Thus,  $441 \nmid 13!$ . If  $n = 26$ ,  $B_K^{26}(3) = 3249 = 9 \times 19^2$ . Hence,  $3249 \nmid 26!$ . So, by Theorem 2, we can prove the nonexistence of a perfect three-error-correcting code in  $S_n$ , where  $n + 1 > 6$  is a prime,  $n^2 + 2n - 6$  has a prime factor  $p > n$ , or  $4 \leq n \leq 33$ .

When  $t = 4$ , by (27), we have  $B_K^n(4) = \frac{(n+1)(n+2)(n^2+3n-12)}{24}$ . First, if  $n + 1 > 6$  or  $n + 2 > 7$  is a prime, then  $B_K^n(4)$  have a prime factor  $p > n$ . Second, we compute  $n^2 + 3n - 12$  for  $5 \leq n \leq 19$  and obtain that  $(n^2 + 3n - 12)(n + 1)(n + 2)$  has a prime factor  $p > n$  except  $n = 13$ . If  $n = 13$ ,  $B_K^{13}(4) = 1715 = 5 \times 7^3$ . Thus,  $1715 \nmid 13!$ . So, by Theorem 2, we can prove the nonexistence of a perfect four-error-correcting code in  $S_n$ , where  $n + 1 > 6$  or  $n + 2 > 7$  is a prime,  $n^2 + 3n - 12$  has a prime factor  $p > n$ , or  $5 \leq n \leq 19$ .

When  $t = 5$ , by (28),  $B_K^n(5) = \frac{(n+7)n(n^3+3n^2-6n-28)}{120}$ . By Theorem 2, we can prove the nonexistence of a perfect five-error-correcting code in  $S_n$ , where  $n + 7 \geq 12$  is a prime or  $n^3 + 3n^2 - 6n - 28$  has a prime factor  $p > n$ .

By the above discussion, we have the following theorem.

**Theorem 3.** *When  $t = 2$ , there are no perfect two-error-correcting codes in  $S_n$ , where  $n + 2 > 6$  is a prime. When  $t = 3$ , there are no perfect three-error-correcting codes in  $S_n$ , where  $n + 1 > 6$  is a prime,  $n^2 + 2n - 6$  has a prime factor  $p > n$ , or  $4 \leq n \leq 33$ . When  $t = 4$ , there are no perfect four-error-correcting codes in  $S_n$ , where  $n + 1 > 6$  or  $n + 2 > 7$  is a prime,  $n^2 + 3n - 12$  has a prime factor  $p > n$ , or  $5 \leq n \leq 19$ . When  $t = 5$ , there are no perfect five-error-correcting codes in  $S_n$ , where  $n + 7 \geq 12$  is a prime or  $n^3 + 3n^2 - 6n - 28$  has a prime factor  $p > n$ .*

## 5 Conclusion

Permutation codes under the Kendall  $\tau$ -metric have been attracted lots of research interest due to their applications in flash memories. In this paper, we considered the nonexistence of perfect codes under the Kendall  $\tau$ -metric. We gave the recursive formulas of the size of a ball or a sphere with radius  $t$  in  $S_n$  under the Kendall  $\tau$ -metric. Specifically, we gave the polynomial expressions of the size of a ball or a sphere with radius  $r$  when  $t = 2, 3, 4$ , or  $5$ . Finally, we used the sphere-packing upper bound to prove that there are no perfect  $t$ -error-correcting codes in  $S_n$  under the Kendall  $\tau$ -metric for some  $n$  and  $t = 2, 3, 4$ , or  $5$ . Specifically, we proved that there are no perfect two-error-correcting codes in  $S_n$ , where  $n + 2 > 6$  is a prime. We also proved that there are no perfect three-error-correcting codes in  $S_n$ , where  $n + 1 > 6$  is a prime,  $n^2 + 2n - 6$  has a prime factor  $p > n$ , or  $4 \leq n \leq 33$ . We further proved that there are no perfect four-error-correcting codes in  $S_n$ , where  $n + 1 > 6$  or  $n + 2 > 7$  is a prime,  $n^2 + 3n - 12$  has a prime factor  $p > n$ , or  $5 \leq n \leq 19$ . We proved that there are no perfect five-error-correcting codes in  $S_n$ , where  $n + 7 \geq 12$  is a prime or  $n^3 + 3n^2 - 6n - 28$  has a prime factor  $p > n$ .

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