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BOUNDS FOR PERMUTATION ARRAYS

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Abstract: A permutation array (P.A.) defined on an r-set of symbols V is a $v \times r$ array of rows each of which is a permutation of the symbols of V and such that any two distinct rows have at most (at least) λ common column entries. We list all known bounds for such arrays and make improvements in certain cases. We consider, at length, the case when every pair of distinct rows of the P.A. have precisely λ common column entries.

1. Introduction

A *permutation array* (P.A.) defined on an r-set of symbols V is a $v \times r$ array such that each row is a permutation of the symbols of V and any two distinct rows of the array have at most (at least) λ common column entries. We denote such an array by $A(r, \leq\lambda; v)$ $(A(r, \geq\lambda; v))$. If every pair of distinct rows in a P.A. have precisely λ common column entries then we call it an *equidistant* permutation array (EPA) and write its parameters as $A(r,\lambda;v)$. The term equidistant is applied since the Hamming distance between rows of an EPA is a constant $r-\lambda$.

In Section 2 we consider EPA's at some length. Section 3 deals with the general permutation array problem.

2. Equidistant permutation arrays

Bolton [1] defines $R(r,\lambda)$ to be the maximum value of v such that an $A(r,\lambda;v)$ exists. In this section we tabulate the known values and bounds for $R(r,\lambda)$.

2.1. Lower bounds

Bolton [1] has shown that

$$R(r,\lambda)\geqq 2+\left\lfloor \lambda\Big/\left\lceil \frac{n}{3}\right\rceil\right\rfloor \quad (n=r-\lambda), \qquad R(r,r-3)=r-1 \tag{1}$$

where $\lfloor x\rfloor$ is the greatest integer less than or equal to x and $\lceil x\rceil$ is the least integer greater than or equal to x. Vanstone [18] has shown that (1) holds with equality whenever $\lambda \geqq \lambda_0(r-\lambda)$. (1) provides a general lower bound for EPA's. This bound is not a good bound when λ is small with respect to $r-\lambda$. In special cases, this bound has been improved. We now consider these special cases.

Woodall [21], using a complete set of pairwise orthogonal latin squares obtained the following result.

Theorem 2.1. *For n a prime or a prime power,*

$$R(4n, n) \geqq n(n-1). \tag{2}$$

An immediate generalization of this result is now given.

Theorem 2.2. *If there exist k pairwise orthogonal Latin squares of order n then*

$$R(4n, n) \geqq kn. \tag{3}$$

Another generalization of Theorem 2.1 was obtained in [19]. In this paper, EPA's are constructed from finite projective geometries. Theorem 2.1 is the special case when we are looking at finite projective planes. The paper also generalizes the notion of a complete set of pairwise orthogonal latin squares. We state the bound obtained in this case as Theorem 2.3.

Theorem 2.3. *For q a prime and n a positive integer,*

$$R\left(\frac{3q(q^{n-1}-1)}{q-1}+q,\ \frac{3q(q^{n-2}-1)}{q-1}+q\right) \geqq (q-1)q^{n-1}. \tag{4}$$

When $n=2$, it is clear that Theorem 2.3 reduces to Theorem 2.1.

Heinrich, van Rees and Wallis [9] have given a lower bound for certain values of r and λ.

Theorem 2.4. *If there exists a set of $\lambda+1$ mutually orthogonal Latin squares of order n, with two disjoint common transversals, then*

$$R(n+\lambda+2, \lambda) \geqq (\lambda+1)n+1. \tag{5}$$

In the case where (3) and (5) can be compared, (5) is a much better bound. From the above theorem, it can be readily shown that if n is a prime or prime power then

$$R(2n-1,\ n-3) \geqq (n-1)^2. \tag{6}$$

2.2. General upper bounds

EPA's are closely related to a class of combinatorial configurations called (r,λ)-designs. For an account of this relationship, the reader is referred to [5]. Using these results and the fact that there is a good bound on the size of (r,λ)-design, it has been shown [4] that

$$R(r,\lambda)\leq\max\{\lambda+2, n^2+n+1\}\tag{7}$$

where $n=r-\lambda$. This has recently [20] been improved to give

$$R(r,\lambda)\leq\max\left\{2+\left\lfloor\lambda\Big/\left\lceil\frac{n}{3}\right\rceil\right\rfloor, n^2-2n+4, \tfrac{1}{2}(n+2)^2, R(n+1,1)\right\},\tag{8}$$

and was obtained by appealing to results on (r,λ)-designs. As indicated earlier,

$$R(r,\lambda)=2+\left\lfloor\lambda\Big/\left\lceil\frac{n}{3}\right\rceil\right\rfloor \quad \text{whenever } \lambda>\tfrac{1}{3}n^3.\tag{9}$$

The EPA's which obtain this bound have been ([18]) completely characterized.

2.3. *The case* $\lambda=1$

Of special interest is the evaluation of $R(r,1)$. Until recently it was not known whether $R(r,1)\geq 2r$ *for any r*. Schellenberg and Taylor [15] have constructed an $A(13,1;27)$. Using a recursive construction in [15], it is possible to construct an infinite family of EPA's which satisfy

$$R(r,1)\geq 2r.\tag{10}$$

We conjecture that (10) is true for all $r>r_0$, a constant.

An obvious lower bound for $R(r,1)$ is

$$R(r,1)\geq r-1.\tag{11}$$

Deza, Mullin and Vanstone [5] established

$$R(2r-1,1)\geq 2r \quad \text{for } r\geq 4.\tag{12}$$

This followed from the existence of a Room square ([13]) of side $2r-1$ for all $r\geq 4$. Vanstone and Schellenberg have shown that if n is a prime power, then

$$R(n^2+n+2,1)\geq 2n^2+n.\tag{13}$$

All of these lower bounds for $R(r,1)$ have been surpassed by the result of Heinrich and van Rees [8].

$$R(n,1)\geq 2n-4 \quad \text{for } n>5.\tag{14}$$

This was established, for $n \neq 8$, by constructing an $A(n, 1; 2n-4)$ using a self-orthogonal Latin square of order $n-2$. An $A(8, 1; 12)$ was constructed with the aid of a computer.

In certain cases, (14) can be improved. In [9], it is shown that if $n-3$ is a prime number congruent to 1 or 3 modulo 8 and greater than 6, then

$$R(n, 1) \geqq 2n-3. \tag{15}$$

As an upper bound, in the case of $\lambda = 1$, we have the result of Mullin and Nemeth [14]:

$$R(r, 1) \leqq r(r-4) \quad \text{for } r \geqq 5. \tag{16}$$

For $r=5$, (16) is sharp. Table 1 shows known values and lower bounds of $R(r, \lambda)$ for values of r and λ.

2.4. *Special types of EPA's*

We say that an EPA (PA) is *irreducible* if it contains no column all of whose entries are identical. Define $R_1(r, \lambda)$ to be the largest value of v for which there exists an irreducible EPA (PA). Due to the constructions given, the bounds (3), (4), (5) apply to $R_1(r, \lambda)$ also. In [15], it was shown that if $2n-1 \not\equiv 3 \pmod 9$ and if $2n-1 \neq 5$, then

$$R_1(2n+1, 2) \geqq 3n-2. \tag{16}$$

An EPA (PA) is said to be *k-uniform* if every symbol occurs either 0 or k times in every column of the array. Define $R^{(k)}(r, \lambda)$ to be the largest value of v such that there exists a k-uniform $A(r, \lambda; v)$.

Table 1.

$r \backslash \lambda$	1	2	3	4	5	6	7	8	9	10
1	∞									
2	1	∞								
3	2	1	∞							
4	3	2	1	∞						
5	5	4	2	1	∞					
6	10	5	5	2	1	∞				
7	13	10	5	6	2	1	∞			
8	≥ 15		10	5	7	2	1	∞		
9	≥ 15	≥ 16			5	8	2	1	∞	
10						5	9	2	1	∞
11							5	10	2	1
12								6	11	2
13	≥ 27									

It can be shown ([5]) that a Room square is equivalent to a 2-uniform $A(r,1;r+1)$. Hence

$$R^2(r,1)=\begin{cases} r+1 & \text{for } r \text{ odd}, r\neq 3, 5,\\ 0 & \text{otherwise}.\end{cases}$$

In general one can show that

$$R^{(k)}(r,\lambda)=\begin{cases} \dfrac{r(k-1)}{\lambda}+1\\ 0\end{cases} \tag{17}$$

depending on whether or not there exists a certain type of resolvable balanced incomplete block design.

A t-wise balanced permutation array t-PA is a $v\times r$ array defined on a r-set of symbols V such that every row of the array is a permutation of the symbols in V and such that any set of t rows have at most (at least) λ common column entries. Such an array is denoted by $t\text{-}A(r,\leqq\lambda;v)$ ($t\text{-}A(r,\geqq\lambda;v)$).

It should be noted that a $t\text{-}A(r,\leqq\lambda;v)$ need not be a $(t-1)\text{-}A(r,\leqq\lambda;v)$.

Define $t\text{-}R(r,\lambda)$ to be the largest value of v such that a $t\text{-}A(r,\lambda;v)$ exists. The only known bound for this function is

$$t\text{-}R\left(n,\binom{n-1-t}{n-2-t}\right)\geqq n-1. \tag{18}$$

This result can be made more general ([18]).

One last special EPA which we consider is one in which the permutations form a group. We denote such an EPA by $A^*(r,\lambda;v)$ and let $R^*(r,\lambda)$ be the maximum value of v such that an $A^*(r,\lambda;v)$ exists. In this case, we consider only irreducible EPA's. For $\lambda=1$, any $A^*(r,\lambda;v)$ is reductible. The results of this section are based on the work of Iwahori [11].

Let A^* be an $A^*(r,\lambda;v)$ which is irreducible. Let f be the number of orbits in A^*. Then, it is shown in [11] that

(a) $\lambda<f<2\lambda$,
(b) $v=(r-\lambda)/(f-\lambda)$. (19)

In the particular case when $\lambda=2$, we have $f=2$, $v=r-2$ and it is also shown in [11] that A^* as a group is isomorphic to one of

(i) A_4 (alternating group on 4 symbols),
(ii) S_4 (symmetric group),
(iii) A_5 (alternating group on 5 symbols),
(iv) a generalized dihedral group.

3. Permutation arrays

3.1 Bounds for $R(r, \leqq\lambda)$, $R(r, \geqq\lambda)$

Bounds of Sections 3.1 and 3.2 are essentially from [2] unless otherwise specified. Using a partition of the set of rows of an $A(r, \leqq\lambda; v)$ or $A(r, \geqq\lambda; v)$ on $\leqq r$ sets of rows having in the first column the numbers $1, 2, \ldots, r$ one can see the following recursive bounds.

$$R(r, \leqq\lambda) \leqq rR(r-1, \leqq\lambda-1), \qquad R(r, \geqq\lambda) \leqq rR(r-, \geqq\lambda-1). \tag{20}$$

Let us denote by $f(a)$, the number of fixed points of the permutation $a \in S_r$. So any two permutations $a, b \in S_r$ have exactly $f(a^{-1}b)$ common positions. Applying the observation that $a_1 b_1 = a_2 b_2 \to a_2^{-1} a_1 = b_2 b_1^{-1} \to f(a_2^{-1} a_1) = f(b_2 b_1^{-1})$ to the set

$$C = \{ab \mid a \in A(r, \geqq\lambda; v_1), b \in A(r, \leqq\lambda-1; v_2)\},$$

one can see that $|C| = v_1 v_2$, $C \subseteq S_r$ and so

$$R(r, \geqq\lambda) R(r, \leqq\lambda-1) \leqq r!. \tag{21}$$

Now, we give the following evident bound

$$R(r, \geqq\lambda) \geqq (r-\lambda)! \tag{22}$$

and its immediate corollary from (21) is

$$R(r, \leqq\lambda) \leqq r!/(r-\lambda-1)!. \tag{23}$$

Let us denote by $d(a, b) = r - f(a^{-1}b)$, the Hamming distance of the permutations $a, b \in S_r$. D_i will be the number of derangements of i letters ($D_i \sim i!/e$). P_j will be the volume of the sphere of radius j in the metric space S_r; hence,

$$P_j = 1 + \sum_{i=2}^{j} D_i \binom{r}{i} \sim e^{-1} r! \sum_{i=2}^{j} \frac{1}{(r-i)!}.$$

For $r \geqq \lambda + 2$, we denote:

$$T_1(r, \lambda) = \sum_{i=0}^{(r-\lambda)/2} \binom{r}{i} D_i \quad \text{if } r-\lambda \text{ is even,}$$

$$= \sum_{i=0}^{(r-\lambda-1)/2} \binom{r}{i} D_i + \binom{r-1}{(r-\lambda-1)/2} D_{(r-\lambda+1)/2} \quad \text{if } r-\lambda \text{ is odd.}$$

and

$$T_2(r,\lambda)=\sum_{i=0}^{(r-\lambda)/2}\binom{r}{i} \quad \text{if } r-\lambda \text{ is even,}$$

$$=\sum_{i=0}^{(r-\lambda-1)/2}\binom{r}{i}+\binom{r}{(r-\lambda-1)/2} \quad \text{if } r-\lambda \text{ is odd}$$

so

$$T_1(r,\lambda)=P_{(r-\lambda)/2} \quad \text{if } r-\lambda \text{ is even,}$$

$$=P_{(r-\lambda+1)/2}-\binom{r-1}{(r-\lambda+1)/2}D_{(r-\lambda+1)/2} \quad \text{if } r-\lambda \text{ is odd.}$$

It is evident that the sphere S of radius $\frac{1}{2}(r-\lambda)$ in S_r forms an $A(r, \geqq\lambda; v)$ because $d(a,b)\leqq r-\lambda$ for any $a,b\in S$. From $|S|=P_{(r-\lambda)/2}$, it follows that

$$R(r, \geqq\lambda)\geqq P_{(r-\lambda)/2}, \quad \text{if } r-\lambda \text{ is even} \tag{24}$$

which was generalized for any $r-\lambda$ to

$$R(r, \geqq\lambda)\geqq T_1(r,\lambda). \tag{25}$$

As an analogue of the Gilbert bound for codes, we have

$$R(r, \leqq\lambda)\geqq r!/P_{r-\lambda-1}. \tag{26}$$

(This bound is valid for any maximal $A(r, \leqq\lambda; v)$, i.e., such that we can not add any permutation to it.)

Let us denote by $E(c)$, the set of letters effectively moved by the permutation $c\in S_r$.

From the left part of the following inequality,

$$|E(a)\,\Delta E(b)|\leqq d(a,b)\leqq|E(a)\cup E(b)|$$

and from the upper bound $T_2(r,\lambda)$ (proved by Kleitman) for the maximal number of sets E_i with $|E_i\,\Delta E_j|\leqq r-\lambda$ (for cardinality of the symmetric difference), it follows that

$$R(r, \geqq\lambda)\leqq D_{r-\lambda}T_2(r,\lambda). \tag{27}$$

Using (21), we obtain from (22) and (25)

$$R(r, \leqq\lambda)\leqq r!/(r-\lambda-1)!, \tag{28}$$

$$R(r, \leqq\lambda)\leqq r!/T_1(r,\lambda+1). \tag{29}$$

(In the case $r-\lambda$ is even, (29) becomes

$$R(r, \leqq\lambda)\leqq r!/P_{(r-\lambda-1)/2}, \tag{30}$$

which is the permutation analogue of the Rao–Hamming bound in coding theory and which can be obtained directly by the same method of packing S_r by spheres of radius $\frac{1}{2}(r-\lambda-1)$.)

3.2. *Uniform arrays*

We now introduce the idea of a generalized Howell design. This is an $r\times r$ array defined on a set of v symbols V such that every cell of the array contains the empty set or a k-subset of V and such that every element of V is contained precisely once in every column of the array and every pair of distinct elements is contained in at most λ cells of the array. Such an array is denoted $S^{(k)}(r, \leqq\lambda; v)$. This is equivalent to a k-uniform PA, $A^{(k)}(r, \leqq\lambda; v)$. It is clear that if there exist k pairwise orthogonal Latin squares of order r, then

$$R^{(k)}(r, \leqq 1)\geqq kr. \tag{31}$$

If (31) holds, this does not imply the existence of k pairwise orthogonal Latin squares of side r. As an example, consider $R^{(2)}(6, \leqq 1)$. Since there exists a Howell design of side 6

$$R^{(2)}(6, \leqq 1)\geqq 12$$

but there do not exist two pairwise orthogonal Latin squares of side 6. This is the only example known to the authors.

3.3. *Cases of equality for bounds of Section 3.1*

From (23), (30), it follows that

$$R(r, <\lambda)\leqq r!/\max\{(r-\lambda)!,\ T_1(r,\lambda)\}.$$

Let us call the permutation array $A(r, <\lambda; v)$ *sharp*, if $v=r!/(r-\lambda)!$ and *perfect* if $v=r!/T_1(r,\lambda)$.

Any sharp and any perfect PA $A(r, <\lambda; v)$ realizes the number $R(r, <\lambda)$. The group S_r itself is a sharp and perfect $A(r, <r; v)$, $A(r, <r-1; v)$.

(i) *Comparison of* $T_1(r,\lambda)$ *and* $(r-\lambda)!$. One can easily check that for $\lambda<r-2$,

$$T_1(r,\lambda)>(r-\lambda)! \quad \text{for } r\geqq r_0(r-\lambda), \tag{32}$$

(for example, $r_0(r-\lambda=4)=8$, $r_0(r-\lambda=6)=14$), and

$$T_1(r,\lambda)<(r-\lambda)! \quad \text{for } r\geqq r_1(\lambda) \tag{33}$$

so, a perfect PA $A(r, <\lambda; v)$ does not exist for $r \geqq r_1(\lambda)$; it is an analogue of the corresponding situation for perfect binary codes (i.e., cases of equality in the Rao–Hamming upper bound).

We do not know any example of a perfect PA.

(ii) *Sharp* PA's. B is a sharp $A(r, <\lambda; r) \to B$ is an $A(r, <\lambda; r!/(r-\lambda)!) \to$ there exists $C = A(r-1, <\lambda-1; (r-1)!/(r-\lambda)!) \to C$ is a sharp $A(r-1, <\lambda-1; v)$. So a sharp PA corresponds to the case of equality in (20).

In particular, the existence of a sharp $A(r, <\lambda; v) \to$ the existence of a sharp $A(r-\lambda+2, <\lambda; v) \to$ there exists a PG$(2, r-\lambda+2)$. (So for example, a sharp $A(r, <r-4; r)$ does not exist because there does not exist a PG $(2, 6)$).

Inequality $T_1(r,\lambda) < (r-\lambda)!$ is a necessary condition for the existence of a sharp $A(r, <\lambda; v)$. From (32), a sharp $A(r, <\lambda; v)$ does not exist for $r \geqq r_0(r-\lambda)$ (for example, a sharp $A(r, r-4; v)$ does not exist for $r \geqq 7$ and a sharp $A(r, <r-6; v)$ does not exist for $r \geqq 14$).

In [2], it was shown that a PA is a sharp $A(r, <\lambda; v)$ iff its rows form a sharply λ-transitive set of permutations of degree r. A sharp $A(r, <1)$ is equivalent to a Latin square, a sharp $A(r, <2; v)$ is equivalent to a PG$(2, r)$. A sharp $A(r, <1; v)$ exists for any r and a sharp $A(r, <2; v)$ exists for any $r = p^a$.

In the case when a Latin square $A(r, <1; v)$ is a group, it is not necessarily a cyclic regular (i.e., sharply 1-transitive group. For the special case when $\frac{1}{2}(r-\lambda-1) = 2$, or $r = \lambda + 5$ we have for a perfect array $R(r, \leqq \lambda) = T_1(r, \lambda+1) = r!/P_2$.

If we have a perfect array, then the following hold:

(i) $r!/P_2 > (r-\lambda-1)! = 4!$,

(ii) $r!/P_2$ is an integer.

For $r \leqq 20$, (ii) is true only for $r = 6, 11$ and 18. On the other hand (i) is true only for $r \geqq 8$. Hence, candidates for a perfect array are $A(11, \leqq 6; 11!/P_2)$ and $A(18, \leqq 13; 18!/P_2)$. (Note that $11!/p_2 = |M_{11}|90$.)

In the case r odd a sharp $A(r, <3; v)$ is equivalent to the sharply 3-transitive group LF $(r, 2^m)$ (surveyed in [3]). There is a known sharply λ-transitive set for $\lambda \geqq 3$ which is not a sharply λ-transitive group (Jordan proved that a sharply λ-transitive group, except S_r $(\lambda = r, r-1)$, A_r $(\lambda = r-2)$, M_{12} $(\lambda = 5)$, M_{11} $(\lambda = 4)$, do not exist for $\lambda > 3$). For any $\lambda \geqq 1$, there exist a sharply λ-transitive set (which is not a group) of permutations for $r = \infty$. The concept of a sharply λ-transitive set is equivalent to the concept of a finite Minkowski-m-structure of order n $(r = n + m, \lambda = m + 1)$. A Minkowski-0-structure is equivalent to an affine plane AG$(2, r)$, which is equivalent to a PG$(2, r)$. A Minkowski-1-structure is called a Minkowski plane. They are equivalent to a sharp $A(r, <3; v)$; they (and the case $\lambda > 3$) were deeply studied in a series of papers by W. Heise, M. Karzel et al. [10], in the language of Minkowski structures.

(iii) *Group* $A(r, <\lambda; v)$. Let us denote by $A^*(r, <\lambda; v)$ any $A(r, <\lambda; v)$ which is a subgroup of S_r. For example, the Frobenius group is exactly a transitive $A^*(r, <2; v)$, any Zassenhaus group is exactly a 2-transitive $A^*(r, <3; v)$, any Gorenstein–Hughes group is exactly a 3-transitive $A^*(r, <4; v)$. All these groups were characterized. In fact, from the Gorenstein–Hughes theorem, it follows that any λ-transitive $A^*(r, <\lambda+1; v)$, which is not sharply λ-transitive is

(a) for $\lambda = 3$, A_6, S_5, M_{11}, PL$(2, 2^P)$ (p is a prime).

(b) for $\lambda=4$, A_7, S_6, M_{12}.

(c) for $\lambda\geqq 5$, $A_{\lambda+3}$, $S_{\lambda+2}$.

Let us denote $R^*(r, <\lambda-1)=\max v$, such that there exists an $A^*(r, <\lambda, v)$. From (23), we have

$$R^*(r, <\lambda)\leqq R(r, <\lambda)\leqq r!/(r-\lambda)!$$

with equality iff there exists a sharply λ-transitive subgroup of S_r. All sharply λ-transitive groups are known. They are: r (and $r-1$)-transitive S_r, $(r-2)$-transitive A_r, the 5-transitive Mathieu group M_{12}, the 4-transitive Mathieu group M_{11}. Any finite sharply 2-transitive group is known to be isomorphic to the group of transformations $x\to a+mx$ on a finite near-field. Any finite sharply 3-transitive group is known to be isomorphic to the group of transformations $x\to(a+m\cdot x)/(b+n\cdot x)$ where $(+)$ and division are those of a Galois field and $(\cdot)$ is either the field multiplication or a proper near-field multiplication. Zassenhaus determined all finite near-fields (and so all sharply 2-transitive groups) and all finite near-fields which give sharply 3-transitive groups. In the infinite case, there exist only sharply 2- or 3-transitive groups.

Hence, we have

$$\begin{aligned}
&R^*(r, \leqq 0)=r,\\
&R^*(r, \leqq 1)=r(r-1) \quad \text{for } r=p^m,\\
&R^*(r, \leqq 2)=r(r-2)\,(r-1) \quad \text{for } r=p^{m+1},\\
&R^*(11, \leqq 3)=11\cdot 10\cdot 9\cdot 8,\\
&R^*(12, \leqq 4)=12\cdot 11\cdot 10\cdot 9\cdot 8,\\
&R^*(r, \leqq r-3)=\tfrac{1}{2}r!,\\
&R^*(r, \leqq r-1)=R^*(r, \leqq r-2)=r!.
\end{aligned} \tag{34}$$

These are all known examples of sharp PA's which are groups.

Table 2 gives known values of $R(r, \leqq\lambda)$ for $\lambda\leqq 4$, $r\leqq 12$. The exact values correspond to sharply transitive groups (except $R(6,1)$ which was determined by computer). A lower bound for $R(r, \leqq 1)$ is (31). Upper bounds (i.e., strict inequalities) correspond to known cases of nonexistence of sharply transitive sets (for $r-\lambda=5$, it is so, because of the nonexistence of PG(2, 6), for $r=\lambda+4\geqq 7$ and for $r=\lambda+6\geqq 14$, it follows from (32); for $r=\lambda+6$, $9\leqq r<14$ it follows from theorem 14 of [10]; for $r=10$, $\lambda=3$ it follows from the nonexistence of a Minkowski 2-structure of order 8 (W. Heise [10])). For $R(8, \leqq 3)$ and $R(9, \leqq 4)$ we give both upper bounds $r!/T_1(r,\lambda+1)$ and $r!/(r-\lambda-1)!$ We recall that $R(10, \leqq 1)=90\to$ PG(2, 10).

Known values for $R(r, \leqq\lambda)$ are given in Table 2.

We note that $R(6, \leqq 1)$ can be shown to be greater than or equal to 18 by the computer results of [7]. McCarthy [12] established $R(6, \leqq 1)=18$ by an exhaustive computer search. He also showed $R(6, \geqq 2)=24$. Van Rees [21] showed that $R(10, \leqq 1)\geqq 32$.

Table 2.

$r \backslash \lambda$	0	1	2	3	4
2	2	2·1			
3	3	3·2	3·2		
4	4	4·3	4·3·2	4·3·2	
5	5	5·4	5·4·3	5·4·3·2	5·4·3·2
6	6	18	6·5·4	6·5·4·3	6·5·4·3·2
7	7	7·6	?<7·6·5	?<7·6·5·4	7·6·5·4·3
8	8	8·7	8·7·6	?≦1390<8·7·6·5	?<8·7·6·5·4
9	9	9·8	9·8·7	?<9·8·7·6	?≦9808 <9·8·7·6·5
10	10	32≦?≦109	10·9·8	?<10·9·8·7	?<10·9·8·7·6
11	11	11·10	?≦11·10·9	11·10·9·8	?≦11·10·9·8·6
12	12	60≦?≦12·11	12·11·10	?≦12·11·10·9	12·11·10·9·8

(iv) *Equalities for* $R(r, \geqq\lambda)$. From (22), (21) it follows that we have $R(r, \geqq\lambda-1) = (r-\lambda+1)!$ for every pair (r,λ), such that $R(r, \leqq\lambda) = r!/(r-\lambda)!$ so we have $R(r, \geqq\lambda-1) = (r-\lambda+1)!$ for pairs (r,λ) as given in (3).

It is evident that $R(r, \geqq r-1) = 1$, $R(r, \geqq r-2) = 2$. For $\lambda > r-2$, it is shown in [2] that

$$R(r, \geqq\lambda) = T_1(r,\lambda) \quad \text{for } r_0(r-\lambda). \tag{35}$$

Also, there are the following two conjectures

$$\text{(A)} \qquad R(r, \geqq\lambda) = (r-\lambda)! \quad \text{for } r \geqq r_0(\lambda), \tag{36}$$

$$\text{(B)} \qquad R(r, \geqq\lambda) = \max_{\mathscr{A}} |B_{\mathscr{A}}|, \tag{37}$$

where $\mathscr{A}$ is a family of subsets $\{F_i\}$ of $\{1,2,\ldots,r]$ such that $|F_i \cup F_j| \leqq r-\lambda$ and

$$B_{\mathscr{A}} = \{a \in S_r | \{1,2,\ldots,r\} - f(a) \in \mathscr{A}\}.$$

Let us remark that the lower bounds (22), (25) and the upper bound (26) for $R(r, \geqq\lambda)$ are in general not the best possible. For example, $R(r, \geqq\lambda) > (r-\lambda)! > T_1(r,\lambda)$ in the case $r \geqq r_0(q)$, $r-\lambda \leqq \sqrt{r}$ $(1<q<2)$. In [2] it was proved also that $r^{-1}\lambda^*(r) \to \frac{1}{2}$ for $r\to\infty$, where λ^* is any value of λ such that $|R(r, \geqq\lambda) - R(r, \leqq\lambda)|$ is minimal.

3.4. *Bounds for* $R_{\text{minmax}}(r, \leqq\lambda)$

Let us denote by $R_{\text{minmax}}(r, \leqq\lambda)$ the minimal v, such that there exists an $A(r, \leqq\lambda; v)$ which is maximal, (i.e. by adding to this $A(r, \leqq\lambda; v)$ any other permutation we cannot obtain $A(r, \leqq\lambda; v+1)$). We have from (26),

$$R_{\text{minmax}}(r, \leqq\lambda) \geqq r!/P_{r-\lambda-1},$$

and so

$$R_{\text{minmax}}(r, \leqq 1) \geqq 4 \quad \text{for } r \geqq 4,$$

$$R_{\text{minmax}}(r, \leqq 2) \geqq 13 \quad \text{for } r \geqq 7,$$

$$R_{\text{minmax}}(r, \leqq 3) \geqq 53 \quad \text{for } r \geqq 8,$$

$$R_{\text{minmax}}(r, \leqq 4) \geqq 271, 273 \quad \text{for } r = 9, 11.$$

Also

$$R_{\text{maxmin}}(r, \leqq 1) = r! \quad \text{for } r = 2, 3.$$

McCarthy remarked that

$$R_{\text{maxmin}}(r, \leqq 1) \leqq r \quad \text{if } r \text{ is even} \tag{38}$$

(because a cyclic Latin square of even order has no transversals [6]). So, $R_{\text{minmax}}(4, \leqq 1) = 4$, $4 \leqq R_{\text{minmax}}(6, \leqq 1) \leqq 6$.

$R_{\text{minmax}}(r, \leqq 0) = r$, because any Latin rectangle can be embedded in Latin square (Ryser).

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