end-to-end average distortion with the SNR, which was referred to as the distortion SNR exponent $a^{\star}(\eta)$, for given spectral efficiency η. We first derived an upper bound on $a^{\star}(\eta)$ based on an informed transmitter that has instantaneous knowledge of the channel capacity. Then the exponent achievable with a separation based scheme was computed. Finally, we proposed HDA source–channel coding schemes that outperform the separated exponent for all η. Remarkably, the HDA scheme for $\eta > 2\min(M, N)$ was shown to achieve the optimal distortion exponent for general M and N. We also showed how to construct practical space–time coding schemes using diversity–multiplexing tradeoff optimal space–time codes and scalar quantization.

References

[1] E. Biglieri, J. Proakis, and S. Shamai (Shitz), "Fading channels: Information-theoretic and communications aspects," *IEEE Trans. Inf. Theory*, vol. 44, no. 6, pp. 2619–2692, Oct. 1998.

[2] L. Zheng and D. N. C. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.

[3] J. N. Laneman, E. Martinian, G. W. Wornell, J. G. Apostolopoulos, and S. J. Wee, "Comparing application- and physical-layer approaches to diversity on wireless channels," in *Intl. Conf. Communications*, Anchorage, AK, May 2003, pp. 2678–2882.

[4] J. N. Laneman, E. Martinian, G. W. Wornell, and J. G. Apostolopoulos, "Source Channel Diversity for Parallel Channels," *IEEE Trans. Inf. Theory*, vol. 51, no. 10, pp. 3518–3539, Oct. 2005.

[5] T. Holliday and A. J. Goldsmith, "Joint source and channel coding for MIMO systems," in *Proc. 42nd Annu. Allerton Conf. Communications, Control, and Computing*, Monticello, IL, Oct. 2004.

[6] D. Gunduz and E. Erkip, "Source and channel coding for quasi-static fading channels," in *Proc. 39th Asilomar Conf. Signals, Systems and Computes*, Monterey, CA, Oct./Nov. 2005, pp. 18–22.

[7] D. Gunduz and E. Erkip, "Source and channel coding for cooperative relaying," in *Proc. Int. Workshop on Signal Processing Advances for Wireless Communications (SPAWC)*, New York City, Jun. 2005, pp. 950–954.

[8] G. Caire and K. R. Narayanan, "On the SNR exponent of hybrid digital analog space time codes," in *Proc. 43rd Annu. Allerton Conf. Communications, Control and Computing*, Monticello, IL, Oct. 2005.

[9] K. R. Narayanan and G. Caire, "Further results on the SNR exponent of hybrid digital analog space time codes," in *Proc. UCSD Workshop on Information Theory and Its Applications*, San Diego, CA, Jan. 2006.

[10] B. Hochwald and K. Zeger, "Tradeoff between source and channel coding," *IEEE Trans. Inf. Theory*, vol. 43, no. 5, pp. 1412–1424, Sep. 1997.

[11] U. Mittal and N. Phamdo, "Hybrid digital-analog (HDA) joint source-channel codes for broadcasting and robust communications," *IEEE Trans. Inf. Theory*, vol. 48, no. 5, pp. 1082–1102, May 2002.

[12] Z. Reznic, M. Feder, and R. Zamir, "Distortion bounds for broadcasting with bandwidth expansion," in *Proc. Conv. Electrical and Electronics Engineers in Israel*, Dec. 2002, p. 19.

[13] B. P. Dunn and J. N. Laneman, "Characterizing source-channel diversity approaches beyond the distortion exponent," in *Proc. 43rd Annu. Allerton Conf. Communications, Control, and Computing*, Monticello, IL, Sep. 2005.

[14] D. Gunduz and E. Erkip, "Joint source-channel codes for MIMO block fading channels," *IEEE Trans. Inf. Theory*, submitted for publication.

[15] D. Gunduz and E. Erkip, "Distortion exponents of MIMO fading channels," in *Proc. IEEE Information Theory Workshop*, Punta del Este, Uruguay, Jul. 2006, pp. 694–698.

[16] M. Gastpar, B. Rimoldi, and M. Vetterli, "To code, or not to code: Lossy source-channel communication revisited," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1147–1158, May 2003.

[17] J. Proakis and M. Salehi, *Fundamentals of Communication Systems*. Upper Saddle River, NJ: Prentice-Hall, 2004.

[18] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.

[19] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*. Berlin/ New York: Springer-Verlag, 1998.

[20] T. Holliday and A. J. Goldsmith, "Optimizing end-to-end distortion in MIMO systems," in *Proc. IEEE Symp. Information Theory*, Adelaide, Australia, Sep. 2005, pp. 1671–1675.

[21] G. Caire and D. Tuninetti, "The throughput of hybrid-ARQ protocols for the Gaussian collision channel," *IEEE Trans. Inf. Theory*, vol. 47, no. 5, pp. 1971–1988, Jul 2001.

[22] H. El Gamal, G. Caire, and M. O. Damen, "The diversity-multiplexing-delay tradeoff in MIMO ARQ channels," in *Proc. IEEE Int. Symp. Information Theory*, Adelaide, Australia, Sep. 2005, pp. 1823–1826.

[23] H. El Gamal, G. Caire, and M. Damen, "Lattice coding and decoding achieve the optimal diversity-vs-multiplexing tradeoff of MIMO channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 6, pp. 968–985, Jun. 2004.

[24] P. Elia, B. Sethuraman, and P. Kumar, "Perfect space-time codes with minimum and nonminimum delay for any number of antennas," in *Int. Conf. Wireless Networks, Communications and Mobile Computing*, Maui, HI, Jun 2005, vol. 1, pp. 722–727.

[25] P. Elia, K. Kumar, S. Pawar, P. Kumar, and H.-F. Lu, "Explicit space-time codes that achieve the diversity-multiplexing gain tradeoff," in *Proc. IEEE Int. Symp. Information Theory*, Adelaide, Australia, Sep. 2005, pp. 896–900.

[26] J.-C. Belfiore, G. Rekaya, and E. Viterbo, "The Golden code: A 2×2 full-rate space-time code with nonvanishing determinant," *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1432–1436, Apr. 2005.

[27] S. Tavildar and P. Viswanath, "Permutation codes: Achieving the diversity-multiplexing tradeoff," in *Proc. IEEE Int. Symp. Information Theory*, Chicago, IL, Jun./Jul. 2004, p. 98.

[28] R. M. Gray, *Source Coding Theory*. Berlin/ew York: Springer-Verlag, 1989.

A Note on the Optimality of Variant-I Permutation Modulation Codes

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***Abstract*—In this correspondence, the optimality of variant-I permutation codes initially proposed by Slepian is shown in a simple way.**

***Index Terms*—Group codes, permutation codes, permutation modulation.**

I. Introduction

In [1], two variants of permutation modulation (PM) codes are introduced. A variant-I code C is defined by l integers $m_1 \leq m_2 \leq \cdots \leq m_l$ satisfying $\sum_{i=1}^{l} m_i = n$ and the n-dimensional vector

$$\boldsymbol{x} = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_l, \ldots, x_l) \tag{1}$$

where for $i = 1, \ldots, l$, x_i appears m_i times. The codebook is formed of all

$$M = \frac{n!}{m_1! \quad m_2! \ldots m_l!} \tag{2}$$

possible distinct permutations of $\boldsymbol{x}$. The rate of the code is $R = \log_2 M/n$ and its minimum squared Euclidean distance (MSED) is defined as

$$d^2_{\min} = \min_{\boldsymbol{x} \neq \boldsymbol{x}' \in C} \|\boldsymbol{x} - \boldsymbol{x}'\|^2. \tag{3}$$

Manuscript received January 18, 2006; revised January 22, 2007.

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Communicated by A. Ashikhmin, Associate Editor for Coding Theory.
Digital Object Identifier 10.1109/TIT.2007.901182

Based on the results of [2], the design of an optimum PM variant-I code can be formulated as follows: "Given the set $\{m_1, m_2, \ldots, m_l\}$, find the corresponding set of points $\{x_1, x_2, \ldots, x_l\}$ such that $d^2_{\min} = 2$ and $\|\boldsymbol{x}\|^2$ is minimized."[1] In fact, [2] provides the solution to this problem but for the last step of the proof, it refers to [3]. In the following, we provide an alternative answer to this part.

In can be noted that the optimum solution derived in [4] is more restrictive as the values m_i's need to satisfy $m_i = m_{l-i-1}$. As a result, many optimum codes found by our approach are not considered by that of [4]. On the other hand, it is straightforward to show that the solution to our optimization design problem for variant-II PM codes is equivalent to that of [4].

II. Optimum Variant-I Codes

In this section, the optimality of variant-I PM codes is derived based on the following.

Lemma 2.1: Consider l integers $m_1 \le m_2 \le \cdots \le m_l$ satisfying $\sum_{i=1}^{l} m_i = n$. Then

$$\frac{1}{n}\sum_{j=1}^{\lfloor l/2 \rfloor} j(m_{l-2j+1} - m_{l-2j}) \le 1/2 \qquad (4)$$

with $m_0 = 0$

Proof: For simplicity of the notations, we assume l is odd, so that $l = 2a - 1$. The case l even follows in a similar way. Proving (4) by contradiction, we assume that

$$\sum_{j=1}^{a-1} j(m_{l-2j+1} - m_{l-2j}) > n/2. \qquad (5)$$

Since

$$m_l + \sum_{j=1}^{a-1}(m_{l-2j+1} + m_{l-2j}) = n, \qquad (6)$$

multiplying (5) by 2 and subtracting (6) from both sides yields, after shifting the pair grouping of consecutive values of m_i's by one

$$\sum_{j=1}^{a-1}(2j-1)(-m_{l-2j+2} + m_{l-2j+1}) - (2a-1)m_1 > 0 \qquad (7)$$

which is a contradiction as all the terms are nonpositive.

A variant-I PM code with $d^2_{\min} = 2$ is optimum if and only if the three following conditions are satisfied (see, e.g., [2] for an equivalent formulation).

- Condition 1: As a set, $\{x_1, \ldots, x_l\} = \{b, b+1, \ldots, b+l-1\}$ for some real value b.
- Condition 2: $\sum_{i=1}^{l} m_i x_i = 0$.
- Condition 3: If $m_i \le m_j$, then $|x_i| \ge |x_j|$.

Condition 1 ensures $d^2_{\min} = 2$ while Condition 2 implies that although $\boldsymbol{x}$ has dimension n, the effective dimension of the code is $n-1$, so that the effective code rate becomes $R = \log_2 M/(n-1)$.

In the following, we present a simple way to achieve these three conditions and for simplicity of the notations, we again assume l is odd; the results are extended in a straightforward way to the case when l is even. For $l = 2a-1$ and $i = 1, \ldots, l$, we assign to each m_i the value $a + k_i$, with $k_i = (-1)^{l-i}\lceil (l-i)/2 \rceil$. In other words, we assign a to m_l, $a-1$ to m_{l-1}, $a+1$ to $m_{l-2}, \ldots, 1$ to m_2, l to m_1. It follows that

$$\frac{1}{n}\sum_{i=1}^{l} m_i(a+k_i) = a - \frac{1}{n}\sum_{j=1}^{a-1} j(m_{l-2j+1} - m_{l-2j}) \qquad (8)$$

with $m_{l-2j+1} - m_{l-2j} \ge 0$.

Defining $x_i = a + k_i - (1/n)\sum_{i=1}^{l} m_i(a+k_i)$, we have

$$x_i = k_i + \frac{1}{n}\sum_{j=1}^{a-1} j(m_{l-2j+1} - m_{l-2j}). \qquad (9)$$

As a result, Conditions 1 and 2 are straightforwardly satisfied. Condition 3 follows from Lemma 2.1, which ensures that the monotonicity of the values $|k_i|$ is preserved.

Note that this result also validates the optimum construction presented in [5] which implicitly assumes that the value a_m has the same number of terms to its left and its right.

Finally, as indicated in Section I, optimum variant II-a and variant II-b codes are obtained in a straighforward way with for $i = 1, \ldots, l$, $x_i = l - i$ and $x_i = l - i + 1/\sqrt{2}$, respectively.

References

[1] D. Slepian, "Permutation modulation," *Proc. IEEE* , vol. 53, no. 3, pp. 228–236, Mar. 1965.

[2] E. Biglieri and M. Elia, "Optimum permutation modulation codes and their asymptotic performance," *IEEE Trans. Inf. Theory*, vol. IT-23, no. 6, pp. 751–753, Nov. 1976.

[3] D. Slepian, "Several new families of alphabets for signaling," Bell Telephone Labs Unpublished Memorandum 1951.

[4] I. Ingemarsson, "Optimized permutation modulation," *IEEE Trans. Inf. Theory*, vol. 36, no. 5, pp. 1098–1100, Sep. 1990.

[5] T. Ericson, INRIA, "Permutation Codes," Rapport de Recherche INRIA 2109, Nov. 1993.

[1] $d^2_{\min} = 2$ allows us to choose integer values for $x_i - x_{i-1}$'s [2].