Improved Permutation Arrays for Kendall-τ Metric

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Definitions:

Let π and σ be permutations on Z_n = {1,2, … ,n}.

An *adjacent transposition* (bubble sort operation) exchanges two adjacent symbols. For example, 1 2 3 4 5 → 1 2 4 3 5 and 1 2 3 4 5 → 2 1 3 4 5.

The *Kendall-τ distance* between π and σ, denoted by d(π, σ), is the minimum number of adjacent transpositions to transform π into σ.

For a set (array) of permutations A, *i.e.* PA, the *distance of A*, denoted d(A), is the minimum Kendall-τ distance between any two permutations in A.

Definitions and Preliminaries:

For positive integers n and d, let *P(n,d)* denote the maximum size of any PA A of permutations on Z_n with distance d.

It is known that, for any n, P(n,1) = n! and P(n,2)=n!/2.

Exact values of P(n,d) are not known generally. Research has focused on obtaining good lower bounds and upper bounds on P(n,d).

Lower Bounds

Theorem 1 (Wang, Zhang, Yang, and Ge; Designs, Codes and Crypto. 2017)

Let m = $\frac{(n-2)^{t+1}-1}{n-3}$, where n-2 is a prime power, then

$$P(n,2t+1) \geq \frac{n!}{(2t+1)m}$$

Examples of Theorem 1:

(a) P(9,7) ≥ 129.6 (We show P(9,7) ≥ 1,008, by a Random/Greedy alg.)

(b) P(9,11) ≥ 1.62 (We show P(9,11) ≥ 101, by a Random/Greedy alg.)

(c) P(7,9) ≥ 14.39 (We show P(7,9) ≥ 16, using an automorphism alg.)

Using automorphisms

It is known that if P is a permutation polynomial (PP) on F_q, *i.e.* P: $F_q \rightarrow F_q$ is a permutation, where F_q is a field of order q, then

(a) Multiplying by a non-zero constant ‘a’, *i.e.* ‘a’ times P(x),

(b) Adding a constant ‘b’ to the argument, *i.e.* P(x+b), and

(c) Adding a constant ‘c’, *i.e.* P(x)+c,

yields another PP.

We use a program to search for representative PPs of equivalence classes defined by combinations of operations (a)-(c). The program finds the largest set of representatives for which the entire class has the stipulated Kendall-τ distance. (This was also done by Buzaglo and Etzion in “Bounds on the size of permutation codes with the Kendall-τ metric”, IEEE Trans. on Info. Theory, 2015. They showed P(7,3) $\geq$588.)

Example

Use operations aP(x)+c on the following 14 representatives found for F_9 at Kendall-τ distance 7:

0 1 2 4 8 3 7 5 6	0 1 2 7 8 5 3 4 6	0 1 3 4 7 2 8 6 5	0 1 3 8 2 6 7 4 5
0 1 3 8 4 6 5 7 2	0 1 4 5 6 7 3 8 2	0 1 4 5 8 2 7 6 3	0 1 6 2 3 4 7 8 5
0 1 6 2 8 7 5 4 3	0 1 6 4 5 2 3 8 7	0 1 6 7 3 4 8 5 2	0 1 7 2 4 6 8 5 3
0 1 7 4 8 3 5 2 6	0 1 8 5 7 4 6 3 2		

Since there are 8 choices for ‘a’ and 9 choices for ‘b’, this yields $8*9*14$=1,008 permutations. Thus, we have P(9,7) $\geq$ 1,008.

Using a Greedy program with randomness

Klϕve, Lin, Tsai, Tzeng in "Permutation arrays under the Chebyshev distance", *IEEE Trans. On Info. Theory*, 2010 described the following Greedy algorithm:

> *Let the identity permutation be the 1st permutation*
>
> *in C. For any set C chosen, choose the next permutation*
>
> *in C to be the lexicographically next permutation in* S_n
>
> *with distance at least d to all in C, if one exists.*

We modified this program to initially choose randomly a specified number of permutations at distance at least d to put into C. We call the program "Random/Greedy". We used Random/Greedy with Kendall-τ distance to get improved lower bounds for P(n,d).

Example:

Using Random/Greedy we found 16 permutations for P(7,9):

2 4 6 7 5 3 1	1 3 6 7 4 5 2	4 5 2 1 7 3 6	6 5 3 2 1 4 7
1 5 4 6 7 3 2	2 3 5 6 7 4 1	5 3 1 7 2 4 6	6 7 5 4 3 1 2
1 2 3 4 5 6 7	3 4 2 7 1 6 5	5 7 2 6 1 4 3	7 3 6 2 1 5 4
1 2 6 7 5 4 3	3 4 5 6 7 1 2	6 4 2 1 3 5 7	7 4 1 6 2 3 5

So, $P(7,9) \geq 16$.

Table: Some Current Lower Bounds for P(n,d)

n\d	3	4	5	6	7	8	9	10	11
5	20	12	6	5	2	2	2	2	
6	102	64	26	20	11	7	4	4	2
7	588	336	126	84	42	28	16	13	8
8	3,752	2,240	672	448	168	115	57	48	26
9	26,831	15,492	3,882	2,497	1,008	608	288	195	101
10	233,421	133,251	29,113	18,344	5,629	3,832	1,489	1,066	492
11	1,330,560	700,263	247,014	153,260	42,013	28,008	9,747	6,890	2,861
12	13,305,600	6,652,800	899,809	595,160	129,298	85,091	73,068	50,649	19,227

Computing Lower Bounds for P(n,d), for larger n and d

To compute a lower bound for a (n,d)-array A, say by a Random/Greedy iterative algorithm, all n! permutations are considered, and, for each one, its distance to every permutation in the current set A is computed.

For example, to compute a (18,15)-array A, this means

18! > 6.4 x 10^{15} permutations + distances.

This is not feasible. We now describe more efficient methods.

Example: To compute a lower bound for P(13,11).

By Theorem 1, with m = $\frac{(11)^6-1}{10} \approx$ 177,166, P(13,2*5+1) $\geq \frac{13!}{11*m} \approx$ 3,195.

Jiang, Schwartz, and Bruck in “Correcting charge-constrained errors in the rank-modulation scheme”, *IEEE Trans. on Info. Theory, 2010*, gave the following:

Theorem 2. For all n,d >1, we have P(n+1,d) $\geq \left\lfloor \frac{n+1}{d} \right\rfloor *$ P(n,d).

This gives P(13,11) $\geq \left\lfloor \frac{13}{11} \right\rfloor *$ P(12,11) ≥ 2*19,227 = 38,454.

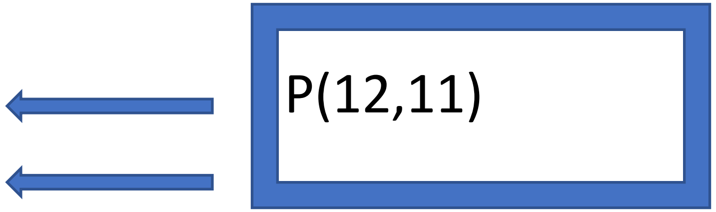
This is good, but we can do better.

Example: To compute a lower bound for P(13,11)   (continued)

(By the previous Theorem 2). Create a (13,11)-PA from two copies of (12,11)-PA:

x 13 x x x x x x x x x x x

x x x x x x x x x x x x 13



Let us generalize:

Let $S_{n,m}$ denote the set of all permutations on Z_n=[1 … n] with the restriction that the first n-m symbols are in sorted order, for any given m < n. A set A $\subseteq S_{n,m}$ with Kendall-τ distance d is called a (n,m,d)-PA or (n,m,d)-array. Let P(n,m,d) be the maximum cardinality of any (n,m,d)-array.

π_1 = x 13 x x x x x x x x x x x

π_2 = x x x x x x x x x x x x 13   is a (13,1,11)-array (with symbols 1-12 replaced by x's)

Example: To compute a lower bound for P(13,11) (continued)

For any permutation π in a (n,m,d)-array A, let P_π(n,d) denote the maximum cardinality of any (n,d)-array with the highest m symbols in the same positions as in π, but where the other n-m symbols can be in any order.

Theorem 3. For any (n,m,d)-array A, P(n,d) $\geq \sum_{\pi \in A} P_\pi$(n,d).

π = x 13 x x x x x x x x x x x ⟸ P_π(13,11) (≥ 31,809)

σ = x x x x x x x x x x x x 13 ⟸ P_σ(13,11). (≥ 19,227)

So, P(13,11) ≥ 51,036. Let us now compute a lower bound for P(14,11).

Example: To compute a lower bound for P(14,11)

By iteration of Theorem 2, P(14,11) $\geq \left\lceil \frac{14}{11} \right\rceil * \left\lceil \frac{13}{11} \right\rceil *$ P(12,11)= 4 $*$ P(12,11) $\geq$ 76,908.

An improvement, using Theorem 3: by a modification of the Random/Greedy program, we computed a (14,2,11)-array with 5 permutations, where the first 12 symbols in each permutation are here replaced by 0's for ease of reading:

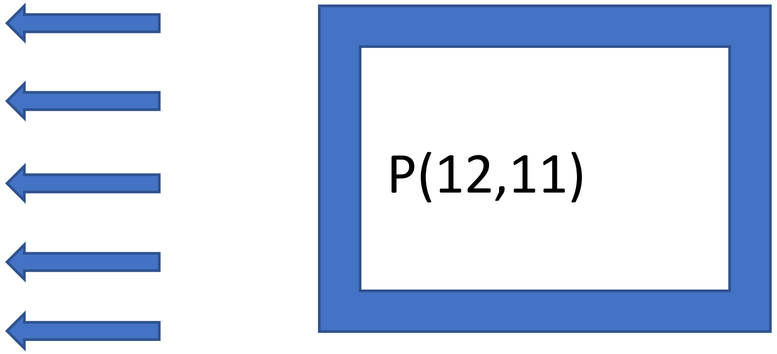
0 0 0 0 0 0 13 0 14 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0 0 0 13 14

0 0 14 0 0 0 0 0 0 0 0 0 0 13

13 14 0 0 0 0 0 0 0 0 0 0 0 0

13 0 0 0 0 0 0 0 0 0 0 0 0 14



Thus, we get P(14,11) $\geq$ 5 $*$ P(12,11). Since, P(12,11) $\geq$ 19,227, P(14,11) $\geq$ 96,135.

Example: To compute a lower bound for P(14,11) (continued)

$\alpha = 0\ 0\ 0\ 0\ 0\ 0\ 13\ 0\ 14\ 0\ 0\ 0\ 0\ 0$ ⟵ $P_\alpha(14,11)$ $(\geq 47{,}851)$

$\beta = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 13\ 14$ ⟵ $P_\beta(14,11)$ $(\geq 19{,}227)$

$\gamma = 0\ 0\ 14\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 13$ ⟵ $P_\gamma(14,11)$ $(\geq 36{,}250)$

$\delta = 13\ 14\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0$ ⟵ $P_\delta(14,11)$ $(\geq 19{,}227)$

$\theta = 13\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 14$ ⟵ $P_\theta(14,11)$ $(\geq 19{,}227)$

So, P(14,11) $\geq$ 141,782.

We can do better.

Example: To compute a lower bound for P(14,11) (continued)

Use a (14,8,11)-array instead of a (14,2,11)-array.

There are, in general, n!/(n-m)! permutations in $S_{n,m}$.

In particular, there are 17,297,280 permutations in $S_{14,8}$. So, this is feasible.

We computed a (14,8,11)-array of 7,909 permutations by a modification of a Random/Greedy algorithm. That is, there is a set A of 7,909 permutations in $S_{14,8}$ with pairwise Kendall-τ distance 11.

Example: To compute a lower bound for P(14,11) (continued)

For each of the 7,909 permutations π in A, compute a lower bound for $P_\pi(14,11)$, denoted by LB($P_\pi(14,11)$).

We computed lower bounds for each $P_\pi(14,11)$, $\pi \epsilon A$, by a modification of a Random/Greedy algorithm. The algorithm takes as input the file A and outputs the sum of { LB($P_\pi(14,11)$) | π in A }

The sum of { LB($P_\pi(14,11)$) | π in A } is 177,098.

So, P(14,11) $\geq$ 177,098.

Example: To compute a lower bound for P(18,15)

By Theorem 1, with m = $\frac{(16)^8-1}{15} \approx 2.86$ x 10^8, P(18,2*7+1) $\geq$ $\frac{18!}{15*m} \approx$ 1,490,669

By computation, P(18,8,15) $\geq$ 9,856. That is, there is a set A of 9,856 permutations in $S_{18,8}$ with pairwise Kendall-τ distance 15.

For each of the 9,856 permutations π in A, compute a lower bound for $P_\pi(18,15)$. The sum of { LB($P_\pi(18,15)$) | π in A } is 19,618,333.

So, P(18,15) $\geq$ 19,618,333.

Additional results

Since P(18,15) $\geq$ 19,618,333, by Theorem 2, *i.e.*

<u>Theorem 2 (Jiang, Schwartz, Bruck)</u>. For all n,d >1, P(n+1,d) $\geq \left\lceil \frac{n+1}{d} \right\rceil * $ P(n,d).

We have P(19,15) $\geq \left\lceil \frac{19}{15} \right\rceil * $ P(18,15) = 2*19,618,333 = 39,236,666.

Whereas, by Theorem 1, *i.e.*

<u>Theorem 1 (Wang, Zhang, Yang, and Ge):</u> Let m = $\frac{(n-2)^{t+1}-1}{n-3}$, where n-2 is a prime power, then P(n,2t+1) $\geq \frac{n!}{(2t+1)m}$.

We have m = $\frac{17^8-1}{16} \approx$ 4.36 x 10^5, and P(19,15) $\geq \frac{19!}{15*m} \approx$ 18,600,815.

Additional theorems

Theorem 4 (Jiang, Schwartz, Bruck) For all n $\geq$ 1 and even d $\geq$ 2,

P(n,d) $\geq \frac{1}{2}\ \mathrm{P}(\mathrm{n}-1,\mathrm{d})$.

Theorem 5 (Jiang, Schwartz, Bruck) For all n,d $\geq$ 1,

P(n+1,d) $\leq$ (n+1)*$\mathrm{P}(\mathrm{n},\mathrm{d})$, $i.e.,\quad \mathrm{P}(\mathrm{n},\mathrm{d}) \geq \frac{P(n+1,d)}{n+1}$

These can also be used to obtain good lower bounds.

Table: Current Lower Bounds for P(n,d)

n\d	5	6	7	8	9	10	11	12	13
12	899,800	595,160	129,298	85,091	73,068	50,649	19,227	13,935	6,087
13	9,363,942	4,681,971	629,301	520,253	236,764	158,208	51,046	29,859	14,158
14		47,638,410	6,522,803	3,693,495	595,728	525,427	178,009	112,338	52,565
15			78,491,859	39,245,930	6,846,611	3,423,306	1,182,803	706,114	190,218
16					33,255,910	18,752,670	8,413,437	4,977,819	1,665,481
17						282,675,240	66,863,784	38,745,418	12,013,962
18									27,520,040

What else can be done?

- One can modify the Random/Greedy algorithm (which is described next).
- One can modify the recursive algorithm, so that one computes good lower bounds for P(n,m,d) by a sequence m_1< m_2< … < m. That is, first compute a (n, m_1,d)-array A. For each $\pi \epsilon A$, compute an (n, m_2,d) array B (for P_π(n, m_2,d)). … Continue the process until obtaining a (n,m,d)-array. This makes it feasible to compute P(n,d) for large n.
- Create a graph whose nodes correspond to permutations π in $S_{n,m}$ and whose edges connect nodes at distance at least d. Assign each node π a weight corresponding to P_π(n,d). Find a maximum weighted clique in this graph to compute a lower bound for P(n,d).

Modifying the Random/Greedy program

Random/Greedy:

Let the identity permutation be the 1st permutation in C. For any set C chosen, choose the next permutation in C to be the lexicographically next permutation in S_n *with distance at least d to all in C, if one exists.*

”Lexicographic” order may not be an obvious choice. For example, consider the order given by the “Steinhaus-Johnson-Trotter” algorithm to enumerate all permutations, where the i^{th} permutation is obtained from the $(i-1)^{th}$ permutation, for all i>1, by a single adjacent transposition.

Example (of SJT order of S_4):

1 2 3 4			Start
1 2 4 3	1 4 2 3	4 1 2 3	'4' moves right-to-left
4 1 3 2			'3' moves left
1 4 3 2	1 3 4 2	1 3 2 4	'4' moves left-to-right
3 1 2 4			'3' moves left
3 1 4 2	3 4 1 2	4 3 1 2	'4' moves right-to-left
4 3 2 1			'2' moves left
3 4 2 1	3 2 4 1	3 2 1 4	'4' moves left-to-right
2 3 1 4			'3' moves right
2 3 4 1	2 4 3 1	4 2 3 1	'4' moves right-to-left
4 2 1 3			'3' moves right
2 4 1 3	2 1 4 3	2 1 3 4	'4' moves left-to-right

Modified Random/Greedy

Modified Random/Greedy:

Let the identity permutation be the 1st permutation
in C. For any set C chosen, choose the next permutation
in C to be next permutation in the SJT sequence
with distance at least d to all in C, if one exists.

There are advantages to this modification. Specifically, if the i^{th} element of the SJT sequence, say π, is put in C, then one can skip the next d-1 permutations, as they are at distance at most d-1 from π.

Thank you for your attention.

Questions?