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Subgroup total perfect codes in Cayley sum graphs

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Abstract

Let Γ be a graph with vertex set V, and let a, b be nonnegative integers. An (a, b)-regular set in Γ is a nonempty proper subset D of V such that every vertex in D has exactly a neighbours in D and every vertex in $V \setminus D$ has exactly b neighbours in D. In particular, a $(1, 1)$-regular set is called a total perfect code. Let G be a finite group and S a square-free subset of G closed under conjugation. The Cayley sum graph $\mathrm{CayS}(G, S)$ of G is the graph with vertex set G such that two vertices x, y are adjacent if and only if $xy \in S$. A subset (respectively, subgroup) D of G is called an (a, b)-regular set (respectively, subgroup (a, b)-regular set) of G if there exists a Cayley sum graph of G which admits D as an (a, b)-regular set. We obtain two necessary and sufficient conditions for a subgroup of a finite group G to be a total perfect code in a Cayley sum graph of G. We also obtain two necessary and sufficient conditions for a subgroup of a finite abelian group G to be a total perfect code of G. We classify finite abelian groups whose all non-trivial subgroups of even order are total perfect codes of the group, and as a corollary we obtain that a finite abelian group has the property that every non-trivial subgroup is a total perfect code if and only if it is isomorphic to an elementary abelian 2-group. We prove that, for a subgroup H of a finite abelian group G and any pair of positive integers (a, b) within certain ranges depending on H, H is an (a, b)-regular set of G if and only if it is a total perfect code of G. Finally, we give a classification of subgroup total perfect codes of a cyclic group, a dihedral group and a generalized quaternion group.

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1 Introduction

All groups considered in this paper are finite, and all graphs considered are finite, undirected and simple. We follow [3] and [12], respectively, for graph- and group-theoretical terminology and notation. As usual, we use $V(\Gamma)$ and $E(\Gamma)$ to denote the vertex set and edge set of a graph Γ, respectively. For a vertex $v \in V(\Gamma)$, the *neighbourhood* of v in Γ, denoted by $N_\Gamma(v)$ or simply $N(v)$, is the set of vertices adjacent to v in Γ. The *degree* of v in Γ, denoted by $\deg(v)$, is the number of edges of Γ incident with v in Γ. The edge between two adjacent vertices u, v is denoted by $\{u, v\}$. The subgraph of Γ *induced* by a subset S of $V(\Gamma)$, denoted by $\Gamma[S]$, is the graph with vertex set S in which two vertices are adjacent if and only if they are adjacent in Γ. Let G be a group with identity element e. If there exists an element y of G such that $x = y^2$, then x is called a *square element*; otherwise x is a *non-square element*. A subset S of G is called a *square-free set* if it contains no square elements of G. A subset S of G is *normal* if it is the union of some conjugacy classes of G, or, equivalently, $g^{-1}Sg := \{g^{-1}sg : s \in S\} = S$ for every $g \in G$. Given a square-free normal subset S of G, the *Cayley sum graph* [1, 5] of G with respect to S, denoted by $\mathrm{CayS}(G, S)$, is the graph with vertex set G such that there is an edge between x to y if and only if $xy \in S$. Since $yx = y(xy)y^{-1}$, the condition that S is normal ensures that $\mathrm{CayS}(G, S)$ is an undirected graph. The condition that S is square-free implies that $\mathrm{CayS}(G, S)$ has no loops. Thus, $\mathrm{CayS}(G, S)$ is an undirected simple $|S|$-regular graph. Given an inverse-closed subset S of $G \setminus \{e\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ is the graph with vertex set G such that there is an edge between x to y if and only if $xy^{-1} \in S$.

Let a and b be nonnegative integers. An *(a, b)-regular set* [4] in a graph Γ is a nonempty proper subset D of $V(\Gamma)$ such that $|N(v) \cap D| = a$ for every $v \in D$ and $|N(v) \cap D| = b$ for every $v \in V(\Gamma)\setminus D$. In particular, a $(0, 1)$-regular set in Γ is called a *perfect code* [11, 17, 22, 24], an *independent perfect dominating set* [10, 15], or an *efficient dominating set* [6, 10]; a $(1, 1)$-regular set in Γ is called a *total perfect code* [11, 27], or an *efficient open dominating set* [7, 10, 19]. For convenience, in the case when Γ is a 1-regular graph we also treat $V(\Gamma)$ as a total perfect code in Γ. Perfect codes, total perfect codes and regular sets in Cayley graphs have been studied extensively in recent years; see, for example, [2, 6, 8, 11, 14, 15, 17, 19–22, 24–27]. In contrast, there are relatively few researches [16, 18, 23] on these sets in Cayley sum graphs. This motivated us to study total perfect codes in Cayley sum graphs in this paper. Similarly to the case of Cayley graphs, a subset D of a group G is called an *(a, b)-regular set* of G if it is an (a, b)-regular set in some Cayley sum graph of G, and an (a, b)-regular set of G is called a *subgroup (a, b)-regular set* of G if it is also a subgroup of G. In particular, a $(0, 1)$-regular set of G is called a *perfect code* of G, and a $(1, 1)$-regular set of G is called a *total perfect code* of G.

The structure and main results in this paper are as follows. In the next section we present some basic results that will be used in subsequent sections. Among other things we give two necessary and sufficient conditions for a subgroup of a group G to be a total perfect code in a given Cayley sum graph of G (Theorem 2.3). In Sect. 3, we first give two necessary and sufficient conditions for a subgroup of an abelian group G to be a total perfect code of G (Theorem 3.2). Using this result, we then classify abelian groups whose all non-trivial subgroups of even order are total perfect codes of the group (Theorem 3.5). As a consequence, we obtain that an abelian group has the property that every non-trivial subgroup is a total perfect code if and only if it is isomorphic to an elementary abelian 2-group (Corollary 3.6). We determine all subgroup total perfect codes of a cyclic group (Theorem 3.7). In Sect. 3, we also prove (Theorem 3.8) that a subgroup H of an abelian group G with $\rho(H) \geq 1$ is a total perfect code of G if and only if it is an (a, b)-regular set of G for any $1 \leq a \leq \rho(H)$ and $1 \leq b \leq |H| - \rho(H)$, where $\rho(H)$ is the number of non-square elements of H. Finally, we determine all subgroup total perfect codes of a dihedral group and a generalized quaternion group (Theorems 4.2 and 4.4) in Sect. 4.

2 A few basic results

As usual, for a group G and a subgroup H of G, we use $|G : H|$ to denote the index of H in G. A *right transversal* of H in G is a subset of G which contains exactly one element from each right coset of H in G. For any two subsets A, B of G, set

$$AB := \{ab : a \in A, b \in B\}.$$

A partition $\pi = \{V_1, V_2, \ldots, V_r\}$ of $V(\Gamma)$ is called an *equitable partition* of a graph Γ (see, for example, [9]) if there is an $r \times r$ matrix $M = (m_{ij})$, called the *quotient matrix* of π, such that for any $1 \leq i, j \leq r$, every vertex in V_i has exactly m_{ij} neighbours in V_j. A graph Γ is called [27] a *pseudocover* of a graph Σ if there exists a surjective mapping $f : V(\Gamma) \to V(\Sigma)$ such that $\Gamma[f^{-1}(v)]$ is a matching for every v in $V(\Sigma)$ and f is a covering projection from Γ^* to Σ, where Γ^* is the graph obtained from Γ by deleting the matching in each $\Gamma[f^{-1}(v)]$. The *fiber* of a vertex or an edge of Σ is its preimage under f.

Observe that if D is a total perfect code in a Cayley sum graph $\mathrm{CayS}(G, S)$ then $|G| = |D||S|$.

Lemma 2.1 *Let G be an abelian group. Let S be a square-free subset of G and set $\Gamma = \mathrm{CayS}(G, S)$. If D is a subset of G such that $D = D^{-1}s$ for some $s \in S$, then $|D|$ is even and $\Gamma[D]$ is a 1-regular subgraph of Γ.*

Proof Since S is a square-free subset of G and $D = D^{-1}s$, every vertex in D has at least one neighbour in $\Gamma[D]$. If there exists a vertex $u \in D$ which is adjacent to two distinct vertices in $\Gamma[D]$, say, $v, w \in D$, then $v^{-1}s = u$ and $u^{-1}s = w$, and hence $vu = s = wu$, a contradiction. Hence $\Gamma[D]$ is a 1-regular subgraph of Γ. Consequently, $|D|$ is even. □

The following result gives a connection between total perfect codes of a Cayley sum graph $\mathrm{CayS}(G, S)$ and partitions of G. This result is similar to [15, Lemma 3(b)] and [27, Lemma 2.1(b)].

Lemma 2.2 *Let G be an abelian group. Let S be a square-free subset of G and set $\Gamma = \mathrm{CayS}(G, S)$. If $D \subseteq G$ is a total perfect code in Γ, then $\{D^{-1}s : s \in S\}$ is a partition of $V(\Gamma) = G$. Conversely, if there is a subset D of G such that $D = D^{-1}s$ for at least one $s \in S$ and $\{D^{-1}s : s \in S\}$ is a partition of G, then D is a total perfect code in Γ.*

Proof Denote $S = \{s_1, s_2, \ldots, s_k\}$. Suppose $D \subseteq G$ is a total perfect code in Γ. Since Γ is $|S|$-regular, we have $|G| = |D||S|$. If $D^{-1}s_i \cap D^{-1}s_j \neq \emptyset$ for two distinct elements s_i, s_j of S, then there exist distinct elements $d_i, d_j \in D$ such that $d_i^{-1}s_i = d_j^{-1}s_j$, which implies that this vertex is adjacent to two distinct vertices in D, a contradiction. Hence

$D^{-1}s_i \cap D^{-1}s_j = \emptyset$ for any two distinct $s_i, s_j \in S$. Combining this with $|G| = |D||S|$, we see that $\{D^{-1}s_1, \ldots, D^{-1}s_k\}$ is a partition of $V(\Gamma) = G$.

Now suppose D is a subset of G such that $D = D^{-1}s_i$ for some i and $\{D^{-1}s_1, \ldots, D^{-1}s_k\}$ is a partition of G. By Lemma 2.1, $\Gamma[D]$ is a 1-regular subgraph of Γ. Moreover, since $\{D^{-1}s_1, \ldots, D^{-1}s_k\}$ is a partition of G, for any vertex $u \in G \setminus D$, there exists a unique element $s_j \in S$ such that $u \in D^{-1}s_j$, which implies that there exists a unique vertex $v \in D$ such that $u = v^{-1}s_j$. That is, every vertex in $G \setminus D$ is adjacent to a unique vertex in D. Hence D is a total perfect code in Γ. □

It is not true that for every total perfect code D in $\mathrm{CayS}(G, S)$ there exists an element $s \in S$ such that $D = D^{-1}s$. So Lemma 2.2 is not a necessary and sufficient condition for a subset D of G to be a total perfect code in $\mathrm{CayS}(G, S)$.

The main result in this section, stated below, is the counterpart of [16, Lemma 2.1] for total perfect codes.

Theorem 2.3 *Let G be a group and H a subgroup of G. Let S be a square-free normal subset of G and set $\Gamma = \mathrm{CayS}(G, S)$. Then the following statements are equivalent:*

(a) *H is a total perfect code in Γ;*
(b) *S is a right transversal of H in G;*
(c) *$|G : H| = |S|$ and $H \cap (H(SS^{-1}\backslash\{e\})) = \emptyset$.*

Proof Note that $V(\Gamma) = G$. Set $S = \{s_1, s_2, \ldots, s_k\}$.

(a)⇒ (b) Since H is a subgroup total perfect code in Γ, by Lemma 2.2, $\{Hs : s \in S\}$ is a partition of G. That is, S is a right transversal of H in G.

(b) ⇒ (a) Suppose that S is a right transversal of H in G. Then $\{Hs_1, Hs_2, \ldots, Hs_k\}$ is a partition of G. Thus, for any vertex $u \in G$, there exists a unique element s_i of S such that $u \in Hs_i$, say, $u = hs_i$ for some $h \in H$. Then u is adjacent to $h^{-1} \in H$ and hence $|N(u) \cap H| \geq 1$. If u is adjacent to two distinct vertices $h_i, h_j \in H$, then $h_i u = s_i, h_j u = s_j$ for two distinct elements $s_i, s_j \in S$, which implies $u = h_i^{-1}s_i = h_j^{-1}s_j \in Hs_i \cap Hs_j$, a contradiction. Hence $|N(u) \cap H| = 1$ for any $u \in G$ and therefore H is a total perfect code in Γ.

(b) ⇒ (c) Suppose that S is a right transversal of H in G. Then $|G : H| = |S|$. If h is an element of $H \cap (H(SS^{-1} \setminus \{e\}))$, then there exists an element $h_i \in H$ such that $h = h_i s_i s_j^{-1}$ for two distinct elements $s_i, s_j \in S$. Thus, $hs_j = h_i s_i$ and $Hs_i = Hs_j$, a contradiction. Therefore, $H \cap (H(SS^{-1}\backslash\{e\})) = \emptyset$.

(c) ⇒ (b) Suppose that $|G : H| = |S|$ and $H \cap (H(SS^{-1}\backslash\{e\})) = \emptyset$. We claim that $Hs_i \cap Hs_j = \emptyset$ for any two distinct elements $s_i, s_j \in S$. Suppose to the contrary that $Hs_i \cap Hs_j \neq \emptyset$ for two distinct elements $s_i, s_j \in S$, then $h_i s_i = h_j s_j$ for some $h_i, h_j \in H$, and hence $h_i = h_j s_j s_i^{-1} \in H \cap (H(SS^{-1} \setminus \{e\}))$, a contradiction. Thus, $Hs_i \neq Hs_j$ for any two distinct elements $s_i, s_j \in S$. Since $|G : H| = |S|$, it follows that S is a right transversal of H in G. □

We now present six corollaries of Theorem 2.3.

Corollary 2.4 *Let G be a group and H a subgroup of G. Let S be a square-free normal subset of G and set $\Gamma = \mathrm{CayS}(G, S)$. If H is a subgroup total perfect code in Γ, then there exists a unique element s of S such that for any edge $\{u, v\}$ of $\Gamma[H]$ we have $vu = uv = s$.*

Proof Since H is a subgroup total perfect code in Γ, by Theorem 2.3, S is a right transversal of H in G. Thus, there exists a unique element s of S such that $H = Hs$. Of course, $s \in H \cap S$.

Since H is a subgroup of G, for any edge $\{u, v\}$ of $\Gamma[H]$, we have $uv, vu \in H \cap S$. Hence $H = H(uv) = H(vu)$. By the uniqueness of s, we then obtain $uv = vu = s$. □

The following result follows from Corollary 2.4 and Theorem 2.3 immediately.

Corollary 2.5 *Let G be a group and H a subgroup of G. Let S be a square-free normal subset of G and set $\Gamma = \mathrm{CayS}(G, S)$. If H is a subgroup total perfect code in Γ, then $|H \cap S| = 1$.*

Corollary 2.6 *Let G be an abelian group and H a subgroup of G. Let S be a square-free subset of G and set $\Gamma = \mathrm{CayS}(G, S)$. If H is a subgroup total perfect code in Γ, then $\{Hs : s \in S\}$ is an equitable partition of Γ, and moreover each $Hs, s \in S$ is a total perfect code in Γ.*

Proof Denote $S = \{s_1, s_2, \dots, s_k\}$. By Theorem 2.3, we know that $\{Hs_1, Hs_2, \dots, Hs_k\}$ is a partition of $V(\Gamma) = G$. We claim that for each t between 1 and k, any vertex $u \in G$ has a unique neighbour in Hs_t. Suppose to the contrary that u has two distinct neighbours vs_t, ws_t in Hs_t, where $v, w \in H$. Then $uvs_t = s_i$ and $uws_t = s_j$ for some $s_i, s_j \in S$. Note that $i \neq j$ as $v \neq w$. Hence $us_t = v^{-1}s_i = w^{-1}s_j \in Hs_i \cap Hs_j$, which contradicts the fact that $\{Hs_1, Hs_2, \dots, Hs_k\}$ is a partition of $V(\Gamma)$. So each vertex of Γ has exactly one neighbour in each part Hs_t. Consequently, $\{Hs_1, Hs_2, \dots, Hs_k\}$ is an equitable partition of Γ whose quotient matrix is the $k \times k$ all-1 matrix, and each part Hs_t is a total perfect code in Γ. □

Combining [27, Lemma 2.5], Theorem 2.3 and Corollary 2.6, we obtain the following result.

Corollary 2.7 *Let G be an abelian group and H a subgroup of G. Let S be a square-free subset of G and set $\Gamma = \mathrm{CayS}(G, S)$. Then the following statements are equivalent:*

(a) *H is a total perfect code in Γ;*
(b) *there exists a pseudocovering $f : \mathrm{CayS}(G, S) \to K_{|S|}$ such that Hs is a vertex fibre of f for some s in S;*
(c) *$\{Hs : s \in S\}$ is an equitable partition of Γ.*

The following result is the counterpart of [16, Proposition 2.5] for total perfect codes.

Corollary 2.8 *Let G be a group and H a normal subgroup of G. Let S be a square-free normal subset of G. If H is a subgroup total perfect code in $\mathrm{CayS}(G, S)$, then for any $g \in G\backslash S$ there exists an element $h \in H\backslash\{e\}$ such that $gh = hg$, and there exists a unique element $h \in (H\backslash\{e\}) \cap S$ such that $gh = hg$.*

Proof Since H is a subgroup total perfect code in $\mathrm{CayS}(G, S)$, by Theorem 2.3, S is a right transversal of H in G. Thus, for any $g \in G \setminus S$, there exists a unique element $s \in S$ such that $g \in Hs$. So $g = hs$ for some $h \in H\backslash\{e\}$. Thus, $Hg = Hs$ and so $Hg^{-1} = Hs^{-1}$. Hence $Hg^{-1}sg = (Hs^{-1})sg = Hg = Hs$. On the other hand, we have $g^{-1}sg \in S$ as S is normal. It follows that $g^{-1}sg = s$ and so $gh = hg$ as $s = h^{-1}g$.

Since both H and S are normal, we have $gHg^{-1} \cap gSg^{-1} = H \cap S$ for each $g \in G$. Moreover, by Corollary 2.5, $H \cap S$ only contains one element, say, h. Since $h \in gHg^{-1} \cap gSg^{-1}$, we have $gh = hg$. Note that $h \in (H\backslash\{e\}) \cap S$ as $e \notin S$. □

Corollary 2.9 *Let G be a group and H a subgroup of G. If H is a subgroup total perfect code of G, then each coset of H in G contains at least one non-square element.*

Proof Since H is a subgroup total perfect code of G, there exists a square-free normal subset S of G such that H is a total perfect code in $\mathrm{CayS}(G, S)$. By Theorem 2.3, for any $g \in G$ there exists an element $s \in S$ such that $g \in Hs$. Hence $Hg = Hs$ and so Hg contains the non-square element s. □

For a subset A of a group G, define

$$\bar{A} = \sum_{g \in G} \mu_A(g) g \in \mathbb{Z}[G],$$

where $\mu_A(g) = 1$ if $g \in A$ and $\mu_A(g) = 0$ if $g \in G \backslash A$. The following is the counterpart of [22, Lemma 2.1] for Cayley sum graphs.

Lemma 2.10 *Let G be a group, D a subset of G, and S a square-free normal subset of G. Let a and b be nonnegative integers. Then the following statements are equivalent:*

(a) *D is an (a, b)-regular set in* $\mathrm{CayS}(G, S)$*;*
(b) *$|D \cap Sg^{-1}| = a$ for each $g \in D$ and $|D \cap Sg^{-1}| = b$ for each $g \in G \backslash D$;*
(c) *$\overline{D^{-1}} \cdot \overline{S} = a\overline{D} + b\overline{G \setminus D}$;*
(d) *$\overline{D^{-1}} \cdot \overline{S} + (b - a)\overline{D} = b\overline{G}$.*

In particular, if D is a subgroup of G, then D is an (a, b)-regular set in $\mathrm{CayS}(G, S)$ *if and only if $|D \cap S| = a$ and $\overline{S \setminus D} \cdot \overline{D} = b\overline{G \setminus D}$.*

Proof Denote $\Gamma = \mathrm{CayS}(G, S)$.

(a) ⇔ (b) This follows from the definition of an (a, b)-regular set in a Cayley sum graph.

(b) ⇔ (c) Note that

$$\begin{aligned}
\overline{D^{-1}} \cdot \overline{S} &= \sum_{d \in D} \sum_{s \in S} d^{-1}s = \sum_{x \in G} \sum_{\substack{(d,s) \in D \times S \\ d^{-1}s = x}} x \\
&= \sum_{x \in G} \sum_{\substack{d \in D \\ dx \in S}} x = \sum_{x \in G} \sum_{\substack{d \in D \\ d \in Sx^{-1}}} x \\
&= \sum_{x \in G} |Sx^{-1} \cap D| x \\
&= \sum_{x \in D} |Sx^{-1} \cap D| x + \sum_{x \in G \backslash D} |Sx^{-1} \cap D| x.
\end{aligned}$$

Note also that (b) holds if and only if

$$\sum_{x \in D} |Sx^{-1} \cap D| x = a\overline{D} \quad \text{and} \quad \sum_{x \in G \backslash D} |Sx^{-1} \cap D| x = b\overline{G \setminus D}.$$

Thus part (b) and part (c) are equivalent.

(c) ⇔ (d) This can be verified by a straightforward computation.

Now assume D is a subgroup of G. Then condition (c) becomes $\overline{D} \cdot \overline{S} = a\overline{D} + b\overline{G \backslash D}$. Since $\{S \cap D, S \backslash D\}$ is a partition of S and $\overline{d} \cdot \overline{D} = \overline{D}$ for all $d \in D$, we have $\overline{D} \cdot \overline{S} = (\overline{S \cap D} + \overline{S \backslash D}) \cdot \overline{D} = \overline{S \cap D} \cdot \overline{D} + \overline{S \backslash D} \cdot \overline{D} = |D \cap S|\overline{D} + \overline{S \backslash D} \cdot \overline{D}$. Hence D is an (a, b)-regular set in Γ if and only if $|D \cap S| = a$ and $\overline{S \backslash D} \cdot \overline{D} = b\overline{G \backslash D}$. □

3 Subgroup total perfect codes of abelian groups

In this section we focus on subgroup total perfect codes of abelian groups. Since any total perfect code contains an even number of vertices, an abelian group cannot admit any subgroup total perfect code unless its order is even. So we only consider abelian groups of even order in this section. Let us begin with the following known result.

Lemma 3.1 [16] *Let $G_1, G_2, \ldots, G_n$ be groups and let H_i be a subgroup of G_i for $1 \leq i \leq n$. If S_i is a right transversal of H_i in G_i for $1 \leq i \leq n$, then $S_1 \times S_2 \times \cdots \times S_n$ is a right transversal of $H_1 \times H_2 \times \cdots \times H_n$ in $G_1 \times G_2 \times \cdots \times G_n$.*

The following is the first main result in this section, where the equivalence between (a) and (b) is the counterpart of [16, Theorem 3.1] for total perfect codes.

Theorem 3.2 *Let G be an abelian group of even order with non-trivial Sylow 2-subgroup P. Let H be a subgroup of G. Then the following statements are equivalent:*

(a) *H is a subgroup total perfect code of G;*
(b) *$H \cap P$ is a subgroup total perfect code of P;*
(c) *H contains a non-square element of G.*

Proof Since P is the Sylow 2-subgroup of G, we may write $G = P \times Q$, where Q is the Hall $2'$-subgroup of G. Since $|G|$ is even, P is non-trivial and hence $H \cap P$ is the Sylow 2-subgroup of H. Let $H = P_1 \times Q_1$, where $P_1 = H \cap P$ and Q_1 consists of the elements of H with odd order.

(a) $\Rightarrow$ (b) Suppose that H is a subgroup total perfect code of G. Then there exists a square-free normal subset S of G such that H is a total perfect code in $\mathrm{CayS}(G, S)$. By Theorem 2.3, S is a right transversal of H in G. Set $l = |G|/|S|$ and denote

$$S = \{(p_1, q_1), \ldots, (p_l, q_l)\},$$

where $p_i \in P$ and $q_i \in Q$ for $1 \leq i \leq l$. Since (p_i, q_i), $1 \leq i \leq l$ are non-square elements of G, we know that p_i, $1 \leq i \leq l$ are non-square elements of P. That is, $T = \{p_1, \ldots, p_l\}$ is a square-free subset of P. Note that $|P_1| \geq 2$, for otherwise each element $(e, q) \in H = \{e\} \times Q_1$ would be a square element, a contradiction. If $P_1 = P$, then $|P_1|$ is even. Since p_i, $1 \leq i \leq l$ are non-square elements of P, there is a non-square element p of P such that P is a total perfect code in $\mathrm{CayS}(P, \{p\})$. Assume $P_1 \neq P$. Define S' to be the subset of T with maximum cardinality such that $P_1 p_i$ for $p_i \in S'$ are pairwise distinct. Then $P_1 p_i$ for $p_i \in S'$ are pairwise disjoint. Since T is a square-free subset of P, so is S'. If $\cup_{p_i \in S'} P_1 p_i$ is a proper subset of P, then there exists $p \in P \backslash (\cup_{p_i \in S'} P_1 p_i)$ such that $p \in P_1 p_t$ for some $p_t \in \{p_1, \ldots, p_l\} \backslash S'$. Thus, $P_1 p_j$ for $p_j \in S' \cup \{p_t\}$ are pairwise distinct, which contradicts the choice of S'. Hence S' is a right transversal of P_1 in P. By Theorem 2.3, it follows that P_1 is a subgroup total perfect code of P.

(b) $\Rightarrow$ (a) Suppose that P_1 is a subgroup total perfect code of P. Take $S' = \{q_0, q_1, \ldots, q_l\}$ to be a right transversal of Q_1 in Q, where $q_0 \in Q_1$. If $P_1 = P$, then there exists a non-square element $x \in P$ such that $S = \{(x, q_0), (x, q_1), \ldots, (x, q_l)\}$ is a square-free subset of G. In this case, S is a right transversal of H in G by Lemma 3.1, and therefore H is a total perfect code of G by Theorem 2.3. Assume $P_1 \neq P$. That is, P is not a subset of H. Since P_1 is a subgroup total perfect code of P, by Theorem 2.3 there exists a square-free subset $S_1 = \{p_1, p_2, \ldots, p_m\}$ of P which is a right transversal of P_1 in P. Define $S = S_1 \times S'$. By Lemma 3.1, S is a right transversal of H in G. Since each $p_i \in S_1$ is a non-square element of

P, each $(p_i, q_j) \in S$ is a non-square element of G. That is, S is a square-free normal subset of G. Thus, by Theorem 2.3, H is a total perfect code in CayS(G, S) and therefore is a total perfect code of G.

(a) $\Rightarrow$ (c) This follows from Corollary 2.9.

(c) $\Rightarrow$ (a) Suppose that H contains a non-square element of G, say, $s \in H$. Then $P_1 \neq \emptyset$ and $|H|$ is even. Since $s \in H$, we may take a right transversal $S = \{s, s_1, \dots, s_k\}$ of H in G containing s. If for some i both $s_i \in S$ and ss_i are square elements, say, $ss_i = g_1^2$, $s_i = g_2^2$ for some $g_1, g_2 \in G$, then $s = g_1^2(g_2^2)^{-1} = (g_1 g_2^{-1})^2$, which contradicts the assumption that s is a non-square element of G. Thus, if $s_i \in S$ is a square element, then ss_i is a non-square element and in this case we replace s_i by ss_i in S. In this way we obtain a square-free subset of G which is also a right transversal of H in G. It follows from Theorem 2.3 that H is a subgroup total perfect code of G. □

By the proof of Theorem 3.2, if a subgroup H of an abelian group G contains a non-square element, then there exists a square-free subset S of G which is a right transversal of H in G. So there is a unique element s of S such that $Hs = H$, and hence $(S \setminus \{s\}) \cup \{e\}$ is also a right transversal of H in G. Combining this with [16, Lemma 2.1], we obtain the following result.

Corollary 3.3 *Let G be an abelian group. Then any a subgroup of G which contains a non-square element is a perfect code of G.*

To prove our second main result in this section we need the following lemma which is analogous to [16, Lemma 3.4].

Lemma 3.4 *Let $G_1, G_2, \dots, G_n$ be cyclic 2-groups. Let $G = G_1 \times G_2 \times \cdots \times G_n$ and let $H = H_1 \times H_2 \times \cdots \times H_n$ be a non-trivial subgroup of G, where H_i is a subgroup of G_i for $1 \le i \le n$. Then H is a subgroup total perfect code of G if and only if $H_i = G_i$ holds for at least one i between 1 and n.*

Proof We first prove the necessity. Suppose that H is a subgroup total perfect code of G. Then there exists a square-free subset $S = \{s_1, s_2, \dots, s_m\}$ of G which is a right transversal of H in G. In particular, $G = Hs_1 \cup Hs_2 \cup \cdots \cup Hs_m$. Note that each element of S is of the form $s_i = (s_{i(1)}, s_{i(2)}, \dots, s_{i(n)})$, where $s_{i(j)}$ is an element of G_j for $1 \le j \le n$. By Corollary 2.5, there exists an element $s_i \in S \cap H$ and an element $s_{i(j)} \in G_j$ such that $s_{i(j)}$ is a non-square element of G_j. Since G_j is a cyclic 2-group and H_j is a subgroup of G_j, we have $\langle s_{i(j)} \rangle = H_j$ and $\langle s_{i(j)} \rangle = G_j$. Hence $H_j = G_j$.

Now we prove the sufficiency. Without loss of generality we may assume $H_1 = G_1$. Since $H_1 = G_1$ is a cyclic 2-group, it contains a non-square element, say, g. Take a right transversal S_i of H_i in G_i for $2 \le i \le n$, and set $S = \{g\} \times S_2 \times \cdots \times S_n$. Since g is a non-square element, each element of S is a non-square element of G. Thus, by Lemma 3.1, S is a right transversal of H in G. Therefore, by Theorem 2.3, H is a subgroup total perfect code of G. □

It would be interesting to determine all groups for which every non-trivial subgroup of even order is a total perfect code. The next result solves this problem for abelian groups (see [16, Theorem 3.5] for a result of the same spirit for subgroup perfect codes of abelian groups).

Theorem 3.5 *Let G be an abelian group. Every non-trivial subgroup of G with even order is a subgroup total perfect code of G if and only if G is isomorphic to $\mathbb{Z}_2^n$ with $n \ge 2$, or $\mathbb{Z}_2^n \times Q$ with Q a non-trivial abelian group of odd order.*

Proof The sufficiency follows from Theorem 3.2 and Lemma 3.4. To prove the necessity, suppose that every non-trivial subgroup of G with even order is a subgroup total perfect code of G. In the case when G is a 2-group, if G has an element g of order 4, then $\langle g^2 \rangle$ contains square elements only and hence is not a total perfect code of G. This contradiction shows that G has no elements of order 4 and so $G \cong \mathbb{Z}_2^n$ for some $n \geq 2$. It remains to consider the case when G is not a 2-group. In this case we have $G = P \times Q$, where P is the Sylow 2-subgroup of G and Q is the Hall $2'$-subgroup of G. Let H be a non-trivial subgroup of G with even order. Then $|H \cap P| \neq 1$. If there is an element g of order 4 in P, then $\langle (e, e), (g^2, e) \rangle$ contains square elements only and hence is not a total perfect code of G, a contradiction. Thus, P does not contain elements of order 4 and hence $G \cong \mathbb{Z}_2^n \times Q$. □

The following corollary follows from Theorem 3.5 immediately.

Corollary 3.6 *Let G be an abelian group. Every non-trivial subgroup of G is a subgroup total perfect code of G if and only if G is isomorphic to $\mathbb{Z}_2^n$ for some $n \geq 2$.*

Cayley sum graphs of $\mathbb{Z}_2^n$ are the same as Cayley graphs of $\mathbb{Z}_2^n$, which are called cubelike graphs in the literature. Since the index of any non-trivial subgroup of $\mathbb{Z}_2^n$ is a power of 2, we see that any cubelike graph admits a subgroup total perfect code if and only if its degree is a power of 2. This result is exactly the first statement in [27, Theorem 4.1].

Using Theorem 3.2, we can construct all subgroup perfect codes of some special abelian groups. For example, from Theorem 3.2 we obtain the following result, which is the counterpart of [16, Theorem 3.7] for total perfect codes.

Theorem 3.7 *Let $G = \langle g \rangle$ be a cyclic group. Let $H = \langle g^t \rangle$ be a non-trivial subgroup of G, where t is the smallest positive integer such that g^t generates H. Then H is a subgroup total perfect code of G if and only if t is odd and $|H|$ is even.*

Proof We first prove the necessity. Suppose H is a subgroup total perfect code of G. Then $|H|$ is even. Also, t must be odd, for otherwise g^{it} is a square element for each $i \geq 1$ and hence, by Corollary 2.5, H is not a subgroup total perfect code of G, a contradiction.

Next we prove the sufficiency. Suppose $t = 2j + 1$ is odd and $|H| = 2i$ is even, where $i, j \geq 1$. Then $g^{2it} = e$ and $t \leq n/2$ by the choice of t. It is readily seen that $S = \{g, g^2, \ldots, g^t\}$ is a right transversal of H in G. If $g^k \in S$ is a square element, then we replace it by g^{k+t} in S. In this way we obtain $S' = \{g, g^{2+t}, g^3, g^{4+t}, \ldots, g^{2j+t}, g^t\}$ which is a square-free subset of G. Since S is a right transversal of H in G, we have $Hg^{2i+1} \neq Hg^{2j+1}$ and $Hg^{2i+t} \neq Hg^{2j+t}$ for $0 \leq i \neq j \leq (t-1)/2$. If $Hg^{2i+1} = Hg^{2j+t}$ for some $0 \leq i, j \leq (t-1)/2$, then $Hg^{2i+1} = Hg^{2j+t} = Hg^t g^{2j} = Hg^{2j}$, but this contradicts the fact that S is a right transversal of H in G. Therefore, S' is a right transversal of H in G. Thus, by Theorem 3.2, H is a subgroup total perfect code of G. □

It follows from Lemma 3.4 and Theorem 3.7 that for any $k \geq 1$ the cyclic group of order 2^k has no non-trivial subgroup total perfect code.

The following result is the counterpart of [22, Theorem 1.2] for Cayley sum graphs. Denote by $\rho(G)$ the number of non-square elements of a group G. Note that $\rho(G) < |G|$ as the identity element of G is a square element.

Theorem 3.8 *Let G be an abelian group and H a subgroup of G with $\rho(H) \geq 1$. Then H is an (a, b)-regular set of G for every pair of integers (a, b) with $1 \leq a \leq \rho(H)$ and $1 \leq b \leq |H| - \rho(H)$ if and only if H is a total perfect code of G.*

Proof Suppose that H is an (a, b)-regular set of G for any $1 \leq a \leq \rho(H)$ and $1 \leq b \leq |H| - \rho(H)$. Since $1 \leq \rho(H) < |H|$, we can take $a = b = 1$ and thus obtain that H is a subgroup total perfect code of G.

Now suppose that H is a total perfect code of G. Then there is a square-free subset $S = \{s_1, s_2, \ldots, s_m\}$ of G such that H is a total perfect code in $\mathrm{CayS}(G, S)$. By Theorem 2.3, S is a right transversal of H in G. Without loss of generality we may assume $H = Hs_1$. Consider an arbitrary pair of integers (a, b) with $1 \leq a \leq \rho(H)$ and $1 \leq b \leq |H| - \rho(H)$. Since H contains $\rho(H)$ non-square elements and $1 \leq a \leq \rho(H)$, we can take a distinct non-square elements $s_{1,1}, s_{1,2}, \ldots, s_{1,a}$ in H. Denote $S_1 = \{s_{1,1}, s_{1,2}, \ldots, s_{1,a}\}$. By Theorem 2.3, for any element $g \in G \setminus H$, there is a unique element $s_i \in S$ such that $g \in Hs_i$. Since H contains $|H| - \rho(H)$ square elements and $1 \leq b \leq |H| - \rho(H)$, we can take b square elements $h_1, h_2, \ldots, h_b$ in H. Since S is a non-square subset of G, we see that $S_2 = \{h_i s_j : 1 \leq i \leq b, 2 \leq j \leq m\}$ is a non-square subset of G. Since S_1 is also a non-square subset of G, so is $S' = S_1 \cup S_2$. We claim that H is an (a, b)-regular set in $\mathrm{CayS}(G, S')$. In fact, since $S_1 \subseteq H$ and H is a subgroup of G, for any $g \in H$, there are a distinct elements $h_{1,1}, h_{1,2}, \ldots, h_{1,a}$ of H such that $h_{1,i} g = s_{1,i}$ for $1 \leq i \leq a$. Since $H \cap Hs_i = \emptyset$ for $2 \leq i \leq m$, it follows that $|H \cap S' g^{-1}| = |H \cap S_1 g| = a$. On the other hand, for any $g \in G \backslash H$, there is an element $s_j \in S \backslash \{s_1\}$ such that $g \in Hs_j$. Since $Hs_j = (Hh_i)s_j = H(h_i s_j)$ for $1 \leq i \leq b$, we have $g \in H(h_i s_j)$ for $1 \leq i \leq b$. Hence there are b elements $h_{i,1}, h_{i,2}, \ldots, h_{i,b}$ in H such that $g = h_{i,j}^{-1} h_i s_j$ for $1 \leq i \leq b$. It follows that there are b elements $h_{i,1}, h_{i,2}, \ldots, h_{i,b}$ in H such that $h_{i,j} g = h_i s_j$ for $1 \leq i \leq b$. So $|H \cap S' g^{-1}| = |H \cap S_2 g^{-1}| = b$ for any $g \in G \setminus H$. Thus, by Lemma 2.10, H is an (a, b)-regular set in $\mathrm{CayS}(G, S)$. Therefore, H is an (a, b)-regular set of G for any pair of integers (a, b) with $1 \leq a \leq \rho(H)$ and $1 \leq b \leq |H| - \rho(H)$. □

4 Dihedral groups and generalized quaternion groups

In this section we determine all subgroup total perfect codes of dihedral groups and generalized quaternion groups. Recall that the dihedral group D_{2n} of order $2n$ is defined as

$$D_{2n} = \langle a, b \mid a^n = b^2 = e, bab = a^{-1} \rangle. \tag{1}$$

The subgroups of D_{2n} are the cyclic groups $\langle a^t \rangle$ with t dividing n and the dihedral groups $\langle a^t, a^r b \rangle$ with t dividing n and $0 \leq r \leq t - 1$.

Lemma 4.1 [13, p. 108] *Let D_{2n} be the dihedral group of order $2n \geq 6$ as given in (1).*

(a) *If n is odd, then the conjugacy classes of D_{2n} are: $\{e\}$, $\{a^i, a^{-i}\}$, $\{a^j b : 0 \leq j \leq n-1\}$, where $1 \leq i \leq (n-1)/2$.*
(b) *If n is even, then the conjugacy classes of D_{2n} are: $\{e\}$, $\{a^{\frac{n}{2}}\}$, $\{a^i, a^{-i}\}$, $\{a^{2j} b : 0 \leq j \leq (n/2) - 1\}$, $\{a^{2j+1} b : 0 \leq j \leq (n/2) - 1\}$, where $1 \leq i \leq (n/2) - 1$.*

The following result is the counterpart of [16, Theorem 4.1] for total perfect codes.

Theorem 4.2 *Let D_{2n} be the dihedral group of order $2n \geq 6$ as given in (1), and let H be a subgroup of D_{2n}. Then H is a subgroup total perfect code of D_{2n} if and only if one of the following holds:*

(a) *n is even and H is one of the following:*

(i) $H = \langle a^{\frac{n}{2}} \rangle$ *and* $n/2$ *is odd;*
(ii) $H = \langle a^t, a^r b \rangle$, *where* $0 \le r \le t-1$, *and either* $t = n$ *or* $n/2$ *is odd and* t *is a divisor of* $n/2$;

(b) n *is odd and* $H = \langle a^r b \rangle = \{e, a^r b\}$, *where* $0 \le r \le n-1$.

Proof (a) We prove the necessity first. Suppose that H is a subgroup total perfect code of D_{2n}. Then there exists a square-free normal subset S of D_{2n} such that H is a total perfect code in $\mathrm{CayS}(D_{2n}, S)$. So $|S||H| = |G| = 2n$ and $|H \cap S| = 1$ by Corollary 2.5. Since H is a subgroup of D_{2n}, we have either $H = \langle a^t \rangle$ with t dividing n or $H = \langle a^t, a^r b \rangle$ with t dividing n and $0 \le r \le t-1$.

Consider $H = \langle a^t \rangle$ first, where t is a divisor of n. Since S is a normal subset of D_{2n}, by part (b) of Lemma 4.1, we must have $H \cap S = \{a^{\frac{n}{2}}\}$. Since S is a square-free subset of D_{2n}, we obtain further that $n/2$ is odd. Since $a^{\frac{n}{2}} \in H$ and t divides n, we may assume without loss of generality that t divides $n/2$. (If t does not divide n, then we can choose another divisor t' of n such that $H = \langle a^{t'} \rangle$ and t' divides $n/2$.) Then $|S| = 2t$. Since $t \le n/2$ and S has an element of the form $a^i b$, we have $2t \ge n/2$ by Lemma 4.1. Hence $n/4 \le t \le n/2$. Since t is a divisor of n and $n/2$ is odd, we have $t = n/2$ or $n/3$. However, if $t = n/3$, then t is even as n is even. So all elements of H are square elements, which contradicts the fact that $H \cap S \ne \emptyset$ and S is square-free. Therefore, we have $t = n/2$ and hence $H = \langle a^{\frac{n}{2}} \rangle$.

Assume $H = \langle a^t, a^r b \rangle$ in the sequel, where t divides n and $0 \le r \le n-1$. Since $|H \cap S| = 1$, we have $|\langle a^t \rangle \cap S| = 0$ or 1.

Case 1. $|\langle a^t \rangle \cap S| = 0$.

Since $|H \cap S| = 1$, we have $\alpha_0, \alpha_1 \in \{0, 1\}$ in this case, where $\alpha_0 = |H \cap \{a^{2j} b : 0 \le j \le (n/2) - 1\}|$ and $\alpha_1 = |H \cap \{a^{2j+1} b : 0 \le j \le (n/2) - 1\}|$. If $\alpha_0 = 0$ and $\alpha_1 = 1$, then $H = \langle a^r b \rangle$ and r is odd. If $\alpha_0 = 1$ and $\alpha_1 = 0$, then $H = \langle a^r b \rangle$ and r is even. If $\alpha_0 = \alpha_1 = 1$, then $H = \langle e, a^{r_2 - r_1}, a^{r_1} b, a^{r_2} b \rangle$ for some odd integer r_1 and even integer r_2 with $1 \le r_1, r_2 \le n-1$. Since $|H \cap S| = 1$, we have $r_2 - r_1 = n/2$ and $n/2$ is odd. Thus, $H = \langle a^t, a^r b \rangle$, where $t = n/2$ is odd.

Case 2. $|\langle a^t \rangle \cap S| = 1$.

In this case, we have $\langle a^t \rangle \cap S = \{a^{\frac{n}{2}}\}$ and $n/2$ is odd. Hence t is an odd divisor of $n/2$.

Up to now we have completed the proof of the necessity.

Now we prove the sufficiency. Consider $H = \langle a^{\frac{n}{2}} \rangle$ first, where $n/2$ is an odd integer. Set

$$S = \{a^{\frac{n}{2}}\} \cup \{a^{2j+1} b : 0 \le j \le (n/2) - 1\} \cup \{a^i, a^{-i} : 1 \le i \le (n/2) - 1,\ i \text{ is odd}\}.$$

Then S is a square-free normal subset of D_{2n}. We have

$$\begin{aligned}
\overline{H}\,\overline{S} &= (e + a^{\frac{n}{2}}) \left(a^{\frac{n}{2}} + \sum_{l=0}^{\frac{n-6}{4}} a^{2l+1} + \sum_{l=0}^{\frac{n-6}{4}} a^{-(2l+1)} + \sum_{j=0}^{\frac{n}{2}-1} a^{2j+1} b \right) \\
&= a^{\frac{n}{2}} + \sum_{l=0}^{\frac{n-6}{4}} a^{2l+1} + \sum_{l=0}^{\frac{n-6}{4}} a^{-(2l+1)} + \sum_{j=0}^{\frac{n}{2}-1} a^{2j+1} b \\
&\quad + e + \sum_{l=0}^{\frac{n-6}{4}} a^{2l+1+\frac{n}{2}} + \sum_{l=0}^{\frac{n-6}{4}} a^{-(2l+1)+\frac{n}{2}} + \sum_{j=0}^{\frac{n}{2}-1} a^{2j+1+\frac{n}{2}} b \\
&= a^{\frac{n}{2}} + \sum_{l=0}^{\frac{n-6}{4}} a^{2l+1} + \sum_{l=0}^{\frac{n-6}{4}} a^{-(2l+1)} + \sum_{j=0}^{\frac{n}{2}-1} a^{2j+1} b
\end{aligned}$$

$$
\begin{aligned}
&+e+\sum_{l'=0}^{\frac{n-6}{4}} a^{2l'}+\sum_{l'=0}^{\frac{n-6}{4}} a^{-2l'}+\sum_{j'=0}^{\frac{n}{2}-1} a^{2j'}b \\
&=\overline{D_{2n}}.
\end{aligned}
$$

If $H=\langle a^t, a^r b\rangle$, where $t=n$, then $H=\langle a^r b\rangle$ and $S=\{a^i b : 0\le i\le n-1\}$ is a square-free normal subset of D_{2n} containing $a^r b\in S$. We have

$$
\overline{H}\,\overline{S}=(e+a^r b)\left(\sum_{i=0}^{n-1} a^i b\right)=\sum_{i=0}^{n-1} a^i b+\sum_{i=0}^{n-1} a^r b a^i b=\sum_{i=0}^{n-1} a^i b+\sum_{i=0}^{n-1} a^{r-i}=\overline{D_{2n}}.
$$

It remains to consider $H=\langle a^t, a^r b\rangle$, where $0\le r\le t-1$, $n/2$ is odd, and t is a divisor of $n/2$. Assume $t=2k+1$. Set

$$
S=S_1\cup\{n/2\},
$$

where $S_1=\{a^{2i+1}, a^{-(2i+1)} : 0\le i\le k-1\}$. Since by Lemma 4.1, S_1 and $\{\frac{n}{2}\}$ are normal subsets of D_{2n}, so is S. Since n is even and j is odd for each $a^j\in S$, S is a square-free normal subset of D_{2n}. Denote by O and E the sets of odd integers and even integers between 1 and $t-1$, respectively. Then

$$
\begin{aligned}
\overline{H}\,\overline{S}&=\left(\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)\right)\left(a^{\frac{n}{2}}+\sum_{j=0}^{k-1}(a^{2j+1}+a^{-(2j+1)})\right)\\
&=\left(\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)\right)a^{\frac{n}{2}}+\left(\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)\right)\left(\sum_{j=0}^{k-1}(a^{2j+1}+a^{-(2j+1)})\right)\\
&=\left(\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)\right)a^{\frac{n}{2}}+\left(\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)\right)\left(\sum_{j=0}^{k-1}a^{2j+1}\right)\\
&\quad+\left(\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)\right)\left(\sum_{j=0}^{k-1}a^{-(2j+1)}\right)\\
&=\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)+\sum_{j\in O}\left(a^j\left(\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)\right)\right)\\
&\quad+\sum_{j\in E}\left(a^j\left(\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)\right)\right)\\
&=\sum_{j=1}^{t}\left(a^j\left(\sum_{i=1}^{n/t}(a^{it}+a^{r+it}b)\right)\right)\\
&=\overline{D_{2n}}.
\end{aligned}
$$

In each case above we obtain from Lemma 2.10 that H is a subgroup total perfect code of D_{2n}.

(b) We first prove the sufficiency. Consider the subgroup $H = \langle a^r b\rangle$ of D_{2n}, where $0 \leq r \leq n-1$. Set $S = \{a^j b : 0 \leq j \leq n-1\}$. Then S is a square-free subset of D_{2n}, and by Lemma 4.1, S is normal as well. We have

$$\overline{H}\,\overline{S} = (e + a^r b)\left(\sum_{i=0}^{n-1} a^i b\right) = \sum_{i=0}^{n-1} a^i b + \sum_{i=0}^{n-1} a^r b a^i b = \sum_{i=0}^{n-1} a^i b + \sum_{i=0}^{n-1} a^{r-i} = \overline{D_{2n}}.$$

Setting $a = b = 1$ in Lemma 2.10, we obtain that H is a subgroup total perfect code of D_{2n}.

Now we prove the necessity. If $H = \langle a^t\rangle$ is a subgroup total perfect code of D_{2n}, where t is a divisor of n, then there exists a square-free normal subset S of D_{2n} such that H is a total perfect code in $\mathrm{CayS}(D_{2n}, S)$. By Corollary 2.5, we have $|H \cap S| = 1$. Since $H = \langle a^t\rangle$, the unique element of $H \cap S$ is of the form a^{it}, where $0 \leq i \leq (n/t)-1$. Since S is a normal subset of D_{2n}, by Lemma 4.1, we have $a^{-it} \in H \cap S$, but this contradicts the fact that $|H \cap S| = 1$. Thus, $H = \langle a^t\rangle$ cannot be a total perfect code of D_{2n}. Suppose $H = \langle a^t, a^r b\rangle$ is a subgroup total perfect code of D_{2n}, where t is a divisor of n and $0 \leq r \leq t-1$. Then $|H \cap S| = 1$ by Corollary 2.5. If $a^i \in H \cap S$ for some positive integer i, then $a^{-i} \in H \cap S$ by Lemma 4.1, but this contradicts the fact that $|H \cap S| = 1$. So there is a positive integer i such that $a^{r+it} b \in H \cap S$. By Lemma 4.1, we know that all elements $a^{r+jt} b$ with $0 \leq j \leq (n-1)/2$ are in $H \cap S$. Since $|H\cap S| = 1$, we must have $a^t = e$ and hence $H = \langle a^r b\rangle = \{e, a^r b\}$. □

The generalized quaternion group Q_{4n} of order $4n \geq 8$ is defined as

$$Q_{4n} = \langle a, b \mid a^n = b^2, a^{2n} = e, b^{-1}ab = a^{-1}\rangle. \tag{2}$$

It is known that the subgroups of Q_{4n} are $\langle a^t\rangle$ with t dividing $2n$ and $\langle a^t, a^r b\rangle$ with t dividing $2n$ and $0 \leq r \leq t-1$.

Lemma 4.3 [13, p. 420] *The generalized quaternion group Q_{4n} has precisely $n+3$ conjugacy classes: $\{e\}$, $\{a^n\}$, $\{a^i, a^{-i}\}$, $\{a^{2j} b : 0 \leq j \leq n-1\}$, $\{a^{2j+1} b : 0 \leq j \leq n-1\}$, where $1 \leq i \leq n-1$.*

Similarly to [16, Theorem 5.1], we can determine all subgroup total perfect codes of Q_{4n}.

Theorem 4.4 *Let Q_{4n} be the generalized quaternion group of order $4n \geq 8$ as given in (2), and let H be a subgroup of Q_{4n}. Then H is a subgroup total perfect code of Q_{4n} if and only if n is odd and $H = \{e, a^r b, b^2, a^r b^3\}$ for some $0 \leq r \leq n-1$.*

Proof Suppose that n is odd and $H = \{e, a^r b, b^2, a^r b^3\}$ for some $0 \leq r \leq n-1$. Then $S = \{a^{2j+1} b : 0 \leq j \leq n-1\}$ is a square-free normal subset of Q_{4n}. Since n is odd, we have

$$\begin{aligned}
\overline{H}\,\overline{S} &= (e + a^r b + b^2 + a^r b^3)\left(\sum_{j=0}^{n-1} a^{2j+1} b\right)\\
&= \sum_{j=0}^{n-1} a^{2j+1} b + \sum_{j=0}^{n-1} a^r b a^{2j+1} b + \sum_{j=0}^{n-1} b^2 a^{2j+1} b + \sum_{j=0}^{n-1} a^r b^3 a^{2j+1} b\\
&= \sum_{j=0}^{n-1} a^{2j+1} b + \sum_{j=0}^{n-1} a^{r-(2j+1)} b^2 + \sum_{j=0}^{n-1} a^{2j+1} b^3 + \sum_{j=0}^{n-1} a^{r-(2j+1)}\\
&= \sum_{j=0}^{n-1} a^{2j+1} b + \sum_{j=0}^{n-1} a^{r+n-(2j+1)} + \sum_{j=0}^{n-1} a^{n+(2j+1)} b + \sum_{j=0}^{n-1} a^{r-(2j+1)}
\end{aligned}$$

$$= \sum_{j=0}^{n-1} a^{2j+1}b + \sum_{j=0}^{n-1} a^{r+2j} + \sum_{j=0}^{n-1} a^{2j}b + \sum_{j=0}^{n-1} a^{r-(2j-1)}$$
$$= \overline{Q_{4n}}.$$

Thus, by Lemma 2.10, H is a subgroup total perfect code of Q_{4n}.

We now prove the necessity. Let H be a subgroup total perfect code of Q_{4n}. Then there exists a square-free normal subset S of Q_{4n} such that H is a total perfect code in $\mathrm{CayS}(Q_{4n}, S)$. By Corollary 2.9, we have $|H \cap S| = 1$. We have either $H = \langle a^t \rangle$ with t dividing $2n$ or $H = \langle a^t, a^r b \rangle$ with t dividing $2n$ and $0 \le r \le t-1$. However, if $H = \langle a^t \rangle$, then $H \cap S = \{b^2\}$, which is a contradiction as b^2 is a square element of Q_{4n}. Thus, $H = \langle a^t, a^r b \rangle$ for some t dividing $2n$ and r between 0 and $t-1$. Since S is normal, by Lemma 4.3, if $a^{it} \in \langle a^t \rangle \cap S$ for some i, then $a^{-it} \in \langle a^t \rangle \cap S$. Since $|H \cap S| = 1$, it follows that $|\langle a^t \rangle \cap S| = 0$. Set $\alpha_0 = |H \cap \{a^{2j}b : 0 \le i \le n-1\}|$ and $\alpha_1 = |H \cap \{a^{2j+1}b : 0 \le i \le n-1\}|$. Then $\alpha_0, \alpha_1 \in \{0, 1\}$. If $\alpha_0 = 1$ and $\alpha_1 = 0$, then $a^{2r}b \in H$ for some $0 \le r \le n-1$, and hence $b^2, a^{2r+n}b \in H$. Thus, $a^{2r+n} \in H \cap \{a^{2j+1}b : 0 \le i \le n-1\}$ or $a^{2r+n} \in H \cap \{a^{2j}b : 0 \le i \le n-1\}$, which is a contradiction. So $(\alpha_0, \alpha_1) \ne (1, 0)$. Similarly, $(\alpha_0, \alpha_1) \ne (0, 1)$. If $\alpha_0 = \alpha_1 = 1$, then we must have $H = \{e, a^r b, b^2, a^r b^3\}$ for some odd r with $0 \le r \le n-1$. Moreover, if n is even, then $a^r b$ and $a^r b^3$ are in the same conjugacy class of Q_{4n}, which contradicts the assumption that $\alpha_0 = \alpha_1 = 1$. Thus, n must be odd. □

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Declarations

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