METRICS ON PERMUTATIONS WITH THE SAME PEAK SET

ALEXANDER DIAZ-LOPEZ, KATHRYN HAYMAKER, KATHRYN KEOUGH, JEONGBIN PARK, EDWARD WHITE

Abstract. Let S_n be the symmetric group on the set $\{1, 2, \ldots, n\}$. Given a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$, we say it has a peak at index i if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$. Let $\mathrm{Peak}(\sigma)$ be the set of all peaks of σ and define $P(S; n) = \{\sigma \in S_n \,|\, \mathrm{Peak}(\sigma) = S\}$. In this paper we study the Hamming metric, ℓ_∞-metric, and Kendall-Tau metric on the sets $P(S; n)$ for all possible S, and determine the minimum and maximum possible values that these metrics can attain in these subsets of S_n.

Keywords: permutations, peaks, Hamming metric, Kendall-Tau metric, L-infinity metric

1. Introduction

In this article we look at various sets of permutations and measure maximum and minimum distances between the elements in these sets. Let S_n be the symmetric group, that is, the set of $n!$ symmetries of $\{1, 2, \ldots, n\}$. We write the elements of S_n in one-line notation, so for $\sigma \in S_n$ we write $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ to denote the permutation that sends $1 \to \sigma_1$, $2 \to \sigma_2, \ldots, n \to \sigma_n$. We say σ has a **peak** at position i in $\{2, 3, \ldots, n-1\}$ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$, that is, σ_i is greater than its two neighbors. We define the **peak set** of σ, $\mathrm{Peak}(\sigma)$, to be the set of all indices at which σ has a peak. For example, if $\sigma = 58327164 \in S_8$ then $\mathrm{Peak}(\sigma) = \{2, 5, 7\}$.

We can collect all permutations that have the same peak set S and define

$$P(S; n) = \{\sigma \in S_n \,|\, \mathrm{Peak}(\sigma) = S\}.$$

We can partition S_n as a disjoint union of sets of the form $P(S; n)$ as we range through all possible peak sets S. The main purpose of this article is to describe the maximum and minimum distances for each subset $P(S; n)$ under three different metrics: the Hamming metric, ℓ_∞-metric, and Kendall-Tau metric. We report our results in Proposition 3.2 and Theorems 3.4, 3.5, and 3.6.

Our study was motivated by recent work on peaks of permutations. The sets $P(S; n)$ were first studied by Nyman in [13] to show that sums of permutations with the same peak set form a subalgebra of the group algebra of S_n (over $\mathbb{Q}$). Later, Billey, Burdzy, and Sagan [2] studied the cardinality of the sets $P(S; n)$ and showed

$$|P(S; n)| = 2^{n-|S|-1} p_S(n),$$

where $p_S(n)$ is a polynomial in n known as the peak polynomial of S. The study of these polynomials has led to a flurry of work such as [3, 8, 9, 10, 14].

Permutations can be used to rank a collection of objects or quantities, and different notions of distances between pairs of permutations have been studied extensively [5, 11]. More recent applications of permutations include data representation, for example in flash memory storage. In the context of data representation, the Hamming metric, the ℓ_∞ metric, and the Kendall-Tau metric have all been considered [1, 4, 12].

2. Metrics on S_n

In this section we will formally define the metrics we use to measure the distance between two permutations. First we recall the definition of a metric. Given a set S, a metric d on S is a map $d : S \times S \to [0, \infty)$ such that for $\sigma, \rho, \tau \in S$,

(1) $d(\sigma, \rho) = 0$ if and only if $\sigma = \rho$,
(2) $d(\sigma, \rho) = d(\rho, \sigma)$,
(3) $d(\sigma, \tau) \leq d(\sigma, \rho) + d(\rho, \tau)$.

In this article, we will use three metrics: the *Hamming metric*, *ℓ_∞-metric*, and *Kendall-Tau metric.*

Definition 2.1. Let d_H, denoting the **Hamming metric**, be the map $d_H : S_n \times S_n \to [0, \infty)$ such that $d_H(\sigma, \rho)$ is the number of indices where σ and ρ differ. That is, if $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ and $\rho = \rho_1\rho_2 \ldots \rho_n$ then

$$d_H(\sigma, \rho) = |\{i \mid \sigma_i \neq \rho_i\}|.$$

Let d_ℓ, denoting the **ℓ_∞-metric**, be the map $d_\ell : S_n \times S_n \to [0, \infty)$ such that

$$d_\ell(\sigma, \rho) = \max\{|\sigma_i - \rho_i| \mid 1 \leq i \leq n\}.$$

Let d_K, denoting the **Kendall-Tau metric**, be the map $d_K : S_n \times S_n \to [0, \infty)$ such that $d_K(\sigma, \rho)$ is the number of pairs (i, j) such that $1 \leq i < j \leq n$ and $(\sigma_i - \sigma_j)(\rho_i - \rho_j) < 0$. The pairs (i, j) counted by d_K are called *deranged pairs.*

Example 2.2. Consider $\sigma, \rho \in S_5$ where $\sigma = 14325$ and $\rho = 25314$. Then, σ and ρ differ in four of the five entries, thus $d_H(\sigma, \rho) = 4$. The differences between the indices of σ and ρ are $|1 - 2|, |4 - 5|, |3 - 3|, |2 - 1|, |5 - 4|$, thus $d_\ell(\sigma, \rho) = 1$. Finally, out of the 10 possible pairs (i, j) with $1 \leq i < j \leq 5$, only $(1, 4)$ and $(2, 5)$ satisfy that $(\sigma_i - \sigma_j)(\rho_i - \rho_j) < 0$, hence $d_K(\sigma, \rho) = 2$.

It is worth noting that the Kendall-Tau metric has an alternative description which is helpful in some contexts. For permutations $\sigma, \rho \in S_n$, let $d'_K(\sigma, \rho)$ be the minimum number of swaps of the form $(i, i + 1)$ that transform σ into ρ, that is, $d'_K(\sigma, \rho)$ is the minimum number n such that there exist transpositions $\tau_1, \ldots, \tau_n$ of the form $(i, i + 1)$ with $\tau_n \cdots \tau_1\sigma = \rho$. In Proposition 2.5 we show that $d_K(\sigma, \rho) = d'_K(\sigma, \rho)$, for all $\sigma, \rho \in S_n$. For example, for the permutations $\sigma = 14325$ and $\rho = 25314$ in Example 2.2, we can swap 1 and 2 and then swap 4 and 5 to convert σ into ρ.

We now present two lemmas, one about d_K and one about d'_K, that will be helpful in proving Proposition 2.5.

Lemma 2.3. *The Kendall-Tau metric is right invariant, that is, for any $\sigma, \tau, \alpha \in S_n$, $d_K(\sigma, \rho) = d_K(\sigma\alpha, \rho\alpha)$.*

Proof. Let (i, j) be any pair such that $1 \leq i < j \leq n$. Consider the pair $(\alpha^{-1}(i), \alpha^{-1}(j))$ or $(\alpha^{-1}(j), \alpha^{-1}(i))$, whichever has the first entry greater than the second entry. Without loss of generality, assume it is $(\alpha^{-1}(i), \alpha^{-1}(j))$. Then,

$$(\sigma\alpha_{\alpha^{-1}(i)} - \sigma\alpha_{\alpha^{-1}(j)})(\rho\alpha_{\alpha^{-1}(i)} - \rho\alpha_{\alpha^{-1}(j)}) = (\sigma_i - \sigma_j)(\rho_i - \rho_j). \tag{1}$$

Thus, if (i, j) is a deranged pair for (σ, ρ) then $(\alpha^{-1}(i), \alpha^{-1}(j))$ is a deranged pair for $(\sigma\alpha, \rho\alpha)$. Similarly, if (i, j) is not a deranged pair for (σ, ρ) (meaning $(\sigma_i - \sigma_j)(\rho_i - \rho_j) > 0$) then by Equation (1) neither $(\alpha^{-1}(i), \alpha^{-1}(j))$ nor $(\alpha^{-1}(j), \alpha^{-1}(i))$ are deranged pairs for $(\sigma\alpha, \rho\alpha)$. Since both (σ, ρ) and $(\sigma\alpha, \rho\alpha)$ have the same number of deranged pairs, then $d_K(\sigma, \rho) = d_K(\sigma\alpha, \rho\alpha)$. □

Lemma 2.4. *For $\sigma, \rho, \alpha \in S_n$, the following statements hold:*
(a) $d'_K(\sigma, \rho) = d'_K(\sigma\alpha, \rho\alpha)$,
(b) $d'_K(\sigma, \rho) = d'_K(\rho, \sigma)$,

Proof. Let $\tau_1, \tau_2, \ldots, \tau_n$ be any collection of transpositions of the form $(i, i + 1)$ that transforms σ into ρ, that is, $\tau_n \cdots \tau_2\tau_1\sigma = \rho$. Multiplying by α on both sides we get $\tau_n \cdots \tau_2\tau_1\sigma\alpha = \rho\alpha$, thus $d'_K(\sigma, \rho) \geq d'_K(\sigma\alpha, \rho\alpha)$. Similarly, let $\tau'_1, \tau'_2, \ldots, \tau'_m$ be any collection of transpositions of the form $(i, i + 1)$ that transforms $\sigma\alpha$ into $\rho\alpha$, that is, $\tau'_n \cdots \tau'_2\tau'_1\sigma\alpha = \rho\alpha$. Multiplying by α^{-1} on the right we get $\tau'_n \cdots \tau'_2\tau'_1\sigma = \rho$. Thus, $d'_K(\sigma, \rho) \leq d'_K(\sigma\alpha, \rho\alpha)$, which completes the proof of part (a).

Part (b) follows from the fact that for any collection $\tau_1, \tau_2, \ldots, \tau_n$ of transpositions of the form $(i, i+1)$ we have that if $\tau_n \cdots \tau_1 \sigma = \rho$ then $\sigma = \tau_1 \cdots \tau_n \rho$. Thus, the minimum number of swaps of the form $(i, i+1)$ that transforms σ into ρ is the minimum number of swaps of the form $(i, i+1)$ that transforms ρ into σ. □

We are now ready to prove that both d_K and d'_K are the same metric.

Proposition 2.5. *For $\sigma, \rho \in S_n$, the value $d_K(\sigma, \rho) = d'_K(\sigma, \rho)$.*

Proof. Theorem 1 in [4] shows that for any permutation $\tau \in S_n$, we have $d_K(\tau, e) = d'_K(\tau, e)$, where e is the identity permutation. This, together with Lemmas 2.3 and 2.4, imply

$$d_K(\sigma, \rho) = d_K(e, \rho\sigma^{-1}) = d_K(\rho\sigma^{-1}, e) = d'_K(\rho\sigma^{-1}, e) = d'_K(e, \rho\sigma^{-1}) = d'_K(\sigma, \rho).$$ □

Remark 2.6. The definition of the Kendall-Tau metric varies among different sources, although most often the definitions are equivalent. For example, Diaconis (p.112, [6]) defines the Kendall-Tau distance between permutations π and σ as follows:

$$I(\pi, \sigma) = \text{minimum number of pairwise adjacent transpositions taking } \pi^{-1} \text{ to } \sigma^{-1}.$$

This definition is equivalent to the one we present in Definition 2.1 and $d'_K(\sigma, \rho)$. The same definition appears in [7], and is named after Kendall based on work in the 1930's and beyond [11]. Some ambiguity arises since Kendall defined a metric on rankings, and rankings can be transformed into permutations in two different ways. A non-equivalent definition of Kendall-Tau distance between permutations is often used in rank modulation applications in the area of coding for flash memory storage (see, for example [1]).

For a given set S of permutations, we will consider the pair-wise distances between distinct permutations in the set as well as the maximum and minimum values attained.

Definition 2.7. For a metric d on a set S, let $d(S)$ be the set of positive integers defined as follows:

$$d(S) = \{d(\sigma, \rho) | \sigma, \rho \in S, \sigma \neq \rho\}.$$

We will denote the minimum and maximum of the set $d(S)$ as $\min(d(S))$ and $\max(d(S))$, respectively.

When $S = S_n$, it is straightforward to compute the values of $\min(d(S))$ and $\max(d(S))$ for the Hamming, ℓ_∞, and Kendall-Tau metrics, as we show in Proposition 2.8. In Section 3, we consider the same question for subsets of S_n defined by their common peak set.

Proposition 2.8. *For S_n with $n \geq 2$, the minimum and maximum for each of the three metrics in Definition 2.1 are*

- $\min(d_H(S_n)) = 2$, $\max(d_H(S_n)) = n$
- $\min(d_\ell(S_n)) = 1$, $\max(d_\ell(S_n)) = n - 1$
- $\min(d_K(S_n)) = 1$, $\max(d_K(S_n)) = \binom{n}{2}$.

Proof. For the Hamming metric, the minimum possible value $\min(d_H(S_n))$ is 2 because distinct permutations must differ in at least 2 indices. This minimum distance is achieved by, e.g., the pair $\sigma = 123 \cdots n$ and $\rho = 213 \cdots n$. The maximum distance occurs when all indices of σ and ρ are different, for example, with $\sigma = 12 \cdots n$ and $\rho = 2\,3\, \cdots n\,1$, so $\max(d_H(S_n)) = n$.

For the ℓ_∞-metric, the minimum possible value of $\min(d_\ell(S_n))$ is 1, which is achieved by, e.g., the pair $\sigma = 12 \cdots n$ and $\rho = 213 \cdots n$. The maximum possible value of $\max(d_\ell(S_n))$ would be $n - 1$, which occurs when $\sigma_i = n$ and $\rho_i = 1$ (or vice-versa) for some index i. The pair $\sigma = 12 \cdots n$ and $\rho = n\, n-1\, \cdots 1$ achieves this maximum.

For Kendall-Tau metric, the minimum distance occurs when the least number of pairs (i, j) such that $1 \leq i < j \leq n$ and $(\sigma_i - \sigma_j)(\rho_i - \rho_j) < 0$ is obtained. The least number of pairs (i, j) possible is 1, and this occurs when $\sigma = 12\cdots n$ and $\tau = 213\cdots n$. The maximum distance occurs when all $\binom{n}{2}$ pairs (i,j) with $1 \le i < j \le n$ satisfy $(\sigma_i - \sigma_j)(\rho_i - \rho_j) < 0$. This happens when $\sigma = 12\cdots n$ and $\rho = n\, n-1\, \cdots 1$. □

3. Maximum and minimum distances among permutations with the same peak set

For the remainder of this paper, we will explore the maximum and minimum values of the three metrics described in Definition 2.1 in sets of permutations with the same peak set. Recall that for any set $S \subseteq [n]$ of indices

$$P(S;n) = \{\sigma \in S_n \,|\, \mathrm{Peak}(\sigma) = S\}.$$

We say S is **admissible** if $P(S;n) \neq \emptyset$.

In Proposition 3.2 we explore $\min(d_K(P(S;n)), \min(d_\ell(P(S;n))$, and $\min(d_H(P(S;n))$ for admissible sets S and in Theorems 3.4, 3.5, and 3.6 we explore the equivalent problem for maximum values. The following lemma is useful for subsequent results.

Lemma 3.1 ([15, Lemma 4.4])**.** *Let S be an admissible set and $\sigma \in P(S;n)$. For any $i \in \{2,3,\ldots,n-1\}$, if i and $i+1$ do not appear consecutively in σ then swapping i and $i+1$ creates a permutation σ' with the same peak set as σ, i.e., $\sigma' \in P(S;n)$. If $i=1$ then swapping 1 and 2 will produce a permutation with the same peak set as σ.*

Proposition 3.2. *Given an admissible peak set S and $P = P(S;n)$ for $n \geq 2$, we have*

$$\min(d_H(P)) = 2, \quad \min(d_\ell(P)) = 1, \text{ and } \min(d_K(P)) = 1.$$

Proof. For any set S, by definition we have that $P(S;n) \subseteq S_n$. Hence, by Proposition 2.8 the minimum value of $\min(d_H(P))$ is at least 2 and for both $\min(d_\ell(P))$ and $\min(d_K(P))$ it is at least 1. Hence, it is enough to find pairs of permutation in P that attain these values.

Let S be admissable and σ be any permutation in $P(S;n)$. By Lemma 3.1, swapping 1 and 2 in σ will lead to a permutation σ' with the same peak set as σ. Since σ and σ' only differ in the indices where 1 and 2 are located, $d_H(\sigma,\sigma') = 2$. Using the same σ and σ' we see that $d_\ell(\sigma,\sigma') = 1$ as the only indices in which they differ have entries 1 and 2 and $|2-1| = |1-2| = 1$. Finally, for the Kendall-Tau metric, we get that $d_K(\sigma,\sigma') = 1$ as the only deranged pair between σ and σ' is the pair of indices where 1 and 2 are located. □

We now proceed to explore the maximum values of the metrics when restricted to sets of permutations with the same peak set. Proposition 2.8 bounds the values of $\max(d_H(P(S;n)))$, $\max(d_\ell(P(S;n)))$, and $\max(d_K(P(S;n)))$ by $n, n-1$, and $\binom{n}{2}$, respectively, as $P(S;n) \subseteq S_n$. In the main results of this section, Theorems 3.4, 3.5, and 3.6 we show that these values are not always attained in the sets $P(S;n)$. Throughout the next results, we will use two particular permutations as our starting point to create others.

Definition 3.3. Let $\mathbf{e}$ be the identity permutation $\mathbf{e} = 1\,2\cdots n-1\,n$ and $\mathbf{e}^* = n\,n-1\cdots 2\,1$. For an admissible peak set S, define $\mathbf{e}[S]$ as the permutation obtained by swapping the entries of k and $k+1$ in $\mathbf{e}$ for each $k \in S$. Similarly, let $\mathbf{e}^*[S]$ be the permutation obtained by swapping the entries of $k-1$ and k in $\mathbf{e}^*$, for each $k \in S$. Since any admissible set S has no consecutive entries, these permutations are well-defined as the order of the swaps does not matter. More explicitly, we have that for $i \in \{1,2,\ldots,n\}$,

$$\mathbf{e}[S]_i = \begin{cases} i+1 & \text{if } i \in S \\ i-1 & \text{if } i \in \{s+1 \,|\, s \in S\} \\ i & \text{otherwise,} \end{cases} \text{ and } \mathbf{e}^*[S]_i = \begin{cases} (n+1-i)+1 & \text{if } i \in S \\ (n+1-i)-1 & \text{if } i \in \{s-1 \,|\, s \in S\} \\ n+1-i & \text{otherwise.} \end{cases}$$

For example, for the set $S = \{2,5,7\}$ and S_9, we have $\mathbf{e}[S] = 132465879$ and $\mathbf{e}^*[S] = 897563421$.

Theorem 3.4. *For $n \geq 2$, the maximum Kendall-Tau distance between permutations in $P(S;n)$ is $\binom{n}{2} - 2|S|$.*

Proof. For any pair of permutations σ and ρ in $P(S;n)$, we have that $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ for each $i \in S$, and analogously for ρ. Therefore, the pairs $(i-1,i)$ and $(i,i+1)$ are not deranged pairs for σ, ρ since

$$(\sigma_{i-1} - \sigma_i)(\rho_{i-1} - \rho_i) > 0 \text{ and } (\sigma_i - \sigma_{i+1})(\rho_i - \rho_{i+1}) > 0.$$

Since there are a total of $\binom{n}{2}$ pairs of possible deranged pairs, we have that $d_K(\sigma,\rho) \leq \binom{n}{2} - 2|S|$.

We now consider a pair of permutations that attain the bound $\binom{n}{2} - 2|S|$. First, note that the permutations $\mathbf{e}$ and $\mathbf{e}^*$ are Kendall-Tau distance $\binom{n}{2}$ apart since for all index pairs (i,j) with $1 \leq i < j \leq n$, $\mathbf{e}$ has $(\mathbf{e}_i - \mathbf{e}_j) < 0$ while $\mathbf{e}^*$ has $(\mathbf{e}^*_i - \mathbf{e}^*_j) > 0$. Consider $\mathbf{e}[S]$ and $\mathbf{e}^*[S]$ as defined in Definition 3.3. We claim $d_K(\mathbf{e}[S], \mathbf{e}^*[S]) = \binom{n}{2} - 2|S|$.

Since every pair of indices (i,j) with $1 \leq i < j \leq n$ was a deranged pair for $\mathbf{e}$ and $\mathbf{e}^*$ and we only altered the (consecutive) entries in indices $k-1$ and k in $\mathbf{e}[S]$, and (consecutive) entries in indices k and $k+1$ in $\mathbf{e}^*[S]$ for $k \in S$, then only the pairs $(k-1,k)$, and $(k,k+1)$ might no longer be deranged pairs for $\mathbf{e}[S]$ and $\mathbf{e}^*[S]$. Indeed, for the pair $(k,k+1)$,

$$\begin{aligned}&(\mathbf{e}[S]_k - \mathbf{e}[S]_{k+1})(\mathbf{e}^*[S]_k - \mathbf{e}^*[S]_{k+1}) \\ &\qquad = (k+1-k)((n+1-k)+1-(n+1-(k+1)) = 2 > 0.\end{aligned}$$

Similarly, for the pair $(k-1,k)$ we have

$$\begin{aligned}&(\mathbf{e}[S]_{k-1} - \mathbf{e}[S]_k)(\mathbf{e}^*[S]_{k-1} - \mathbf{e}^*[S]_k) \\ &\qquad = (k-1-(k+1))(n+1-(k-1)-1-((n+1-k)+1)) = 2 > 0.\end{aligned}$$

Thus, for each peak in S we have two pairs that are not deranged, hence $d_K(\mathbf{e}[S], \mathbf{e}^*[S]) = \binom{n}{2} - 2|S|$. □

Theorem 3.5. *For $n \geq 2$, the maximum ℓ_∞ distance between permutations in $P(S;n)$ is $n-2$ when S contains peaks at indices 2 and $n-1$, and $n-1$ otherwise.*

Proof. First consider the case when $\{2, n-1\} \not\subseteq S$. If $2 \notin S$, then the permutations $\mathbf{e}[S]$ and $\mathbf{e}^*[S]$ achieve the maximum ℓ_∞-distance $n-1$ as $\mathbf{e}[S]_1 = 1$ and $\mathbf{e}^*[S]_1 = n$. Similarly, if $n-1 \notin S$ then $\mathbf{e}[S]_n = n$ and $\mathbf{e}^*[S]_n = 1$, and therefore $d_\ell(\mathbf{e}[S], \mathbf{e}^*[S]) = n-1$.

If $\{2, n-1\} \subseteq S$, then we first claim $d_\ell(\sigma,\rho) \leq n-2$ for any pair of distinct permutations $\sigma, \rho \in P(S;n)$. In any permutation in $P(S;n)$, n must not appear in index 1 nor index n since indices 2 and $n-1$ are peaks. Since n is larger than any other number in the permutation, it must be in an index that is a peak. On the other side, 1 will never appear in an index that is a peak. Hence, $d_\ell(\sigma,\rho)$ cannot be $n-1$ as the only way to obtain this would be for n and 1 to appear in the same index in σ and ρ, respectively. Thus, $d_\ell(\sigma,\rho) \leq n-2$. To show that this bound is achieved, consider the permutations $\mathbf{e}[S]$ and $\mathbf{e}^*[S]$. Since $\mathbf{e}[S]_1 = 1$ and $\mathbf{e}^*[S]_1 = n-1$, then $d_\ell(\mathbf{e}[S], \mathbf{e}^*[S]) = n-2$. □

The next result considers the maximum Hamming distance between permutations with the same peak set in S_n for $n \geq 4$. We first remark that for $n=2$, the only peak set is $\emptyset$ and $\max d_H(P(\emptyset;2)) = 2$, and for $n=3$, we have that $\max d_H(P(\emptyset;3)) = 3$ and $\max d_H(P(\{2\};3)) = 2$.

Theorem 3.6. *For $n \geq 4$ and any admissible peak set S, the maximum Hamming distance between permutations in $P(S;n)$ is n.*

Proof. We proceed by induction on n. For the base cases of $n=4$ and $n=5$, consider the pairs of permutations in each of the admissible peak sets shown in Tables 1 and 2, respectively. Suppose that for every $4 \leq j < n$ the maximum Hamming distance between permutations in $P(S;j)$ is j.

$S=\emptyset$	$S=\{2\}$	$S=\{3\}$
1234	1324	1342
4321	2431	4231

TABLE 1. Pairs of permutations in S_4 with the same peak set and Hamming distance four.

$S=\emptyset$	$S=\{2\}$	$S=\{3\}$	$S=\{4\}$	$S=\{2,4\}$
12345	13245	13425	43251	13254
53214	25314	52314	54132	45132

TABLE 2. Pairs of permutations in S_5 with the same peak set and Hamming distance five.

$S=\emptyset$	$S=\{2\}$	$S=\{3\}$	$S=\{4\}$	$S=\{5\}$	$S=\{2,4\}$	$S=\{2,5\}$	$S=\{3,5\}$
123456	132456	134256	432516	123465	132546	132465	134265
632145	263145	623145	641325	632154	461325	263154	623154

TABLE 3. Pairs of permutations in S_6 with the same peak set and Hamming distance six, created using the constructions in the proof of Theorem 3.6.

Let S be an admissible peak set for permutations in S_n, and for this case assume $n-1 \notin S$. Since $n-1 \notin S$, S is also an admissible peak set for permutations in S_{n-1}, so there exist permutations $\sigma, \rho \in P(S; n-1)$ such that $d_H(\sigma, \rho) = n-1$ by the inductive hypothesis. Since σ and ρ differ in every index, in at least one of the permutations $n-1$ does not appear in index $n-1$. Without loss of generality, assume $\rho_{n-1} \neq n-1$. Construct permutations σ', ρ' in S_n as follows: σ' equals σ with n appended at the end. For ρ', first form an intermediate permutation ρ'' by appending n to the end of ρ. Then to obtain ρ', swap values n and $n-1$ in ρ''. We claim that $d_H(\sigma', \rho') = n$ and the peak set of both σ' and ρ' is S.

First recall that since $d_H(\sigma, \rho) = n-1$, we have that $d_H(\sigma', \rho'') = n-1$ since they are formed by appending n to the end of each permutation. Swapping the values $n-1$ and n in ρ'' results in the distance $d_H(\sigma', \rho') = n$. The peak set of σ' and ρ'', S, is inherited from σ and ρ by construction. Since $\rho_{n-1} \neq n-1$ then $n-1$ and n are not neighbors in ρ''. By Lemma 3.1 the peak set of ρ' is the same as the peak set of ρ'', which is S.

Now assume S is an admissible peak set for permutations in S_n, and $n-1 \in S$. Define $S' = S \setminus \{n-1\}$, which is an admissible peak set on S_{n-2}. By our inductive assumption there exist permutations $\sigma, \rho \in P(S'; n-2)$ such that $d_H(\sigma, \rho) = n-2$. Thus, at least one of σ or ρ must have its $n-2$ index not equal to $n-2$. Without loss of generality, suppose $\rho_{n-2} \neq n-2$. Define the following permutations in S_n: σ' equals σ with values n and $n-1$ appended to the end, in that order, that is,

$$\sigma' = \sigma_1 \cdots \sigma_{n-2}\, n\, n-1.$$

Starting with ρ, define ρ'' to be the permutation ρ with n and $n-1$ appended to the end in that order. Let i be the index such that $\rho_i = n-2$, then ρ'' is of the form

$$\rho'' = \rho_1 \cdots \rho_{i-1}\, n-2\, \rho_{i+1} \cdots \rho_{n-2}\, n\, n-1.$$

Let ρ' be

$$\rho' = \rho_1 \cdots \rho_{i-1}\, n\, \rho_{i+1} \cdots \rho_{n-2}\, n-1\, n-2.$$

In other words, ρ' equals ρ with value $n-2$ replaced by n, and then $n-1$, $n-2$ appended in that order to the end of the permutation. By construction, σ' and ρ' differ in every index, so $d_H(\sigma', \rho') = n$. Finally, the peak set of both σ' and ρ' is S as we have introduced a peak at $n-1$ and have not altered any other entry other than cyclically permuting $(n, n-1, n-2)$ in ρ'', which does not change the peak set. Hence, the result is proven. Table 3 showcases these constructions for the case $n = 6$. □

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