Cryptography and Communications
https://doi.org/10.1007/s12095-023-00642-6

RESEARCH

New upper bounds on the size of permutation codes under Kendall τ-metric

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Received: 12 October 2022 / Accepted: 24 March 2023

Abstract
We give two methods that are based on the representation theory of symmetric groups to study the largest size $P(n, d)$ of permutation codes of length n, i.e., subsets of the set S_n of all permutations on $\{1, \ldots, n\}$ with the minimum distance (at least) d under the Kendall τ-metric. The first method is an integer programming problem obtained from the transitive actions of S_n. The second method can be applied to refute the existence of perfect codes in S_n. Applying these methods, we reduce the known upper bound $(n-1)!-1$ for $P(n, 3)$ to $(n-1)! - \lceil \frac{n}{3} \rceil + 2 \leq (n-1)! - 2$, whenever $n \geq 11$ is prime. If $n = 6, 7, 11, 13, 14, 15, 17$, the known upper bound for $P(n, 3)$ is decreased by 3, 3, 9, 11, 1, 1, 4, respectively.

Keywords Rank modulation · Kendall τ-metric · Permutation codes

Mathematics Subject Classification (2010) 94B25 · 94B65 · 68P30

1 Introduction

Rank modulation was proposed as a solution to the challenges posed by flash memory storages [9]. In the rank modulation framework, codes are permutation codes, where by a Permutation Code (PC) of length n we simply mean a non-empty subset of S_n, the set of all permutations of $[n] := \{1, 2, \ldots, n\}$. Given a permutation $\pi := [\pi(1), \pi(2), \ldots, \pi(i), \pi(i+1), \ldots, \pi(n)] \in S_n$, an adjacent transposition, $(i, i+1)$, for some $1 \leq i \leq n-1$, applied to π will result in the permutation $[\pi(1), \pi(2), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n)]$. For two permutations $\rho, \pi \in S_n$, the Kendall τ-distance between ρ and π, $d_K(\rho, \pi)$, is defined as the minimum number of adjacent transpositions needed to write $\rho\pi^{-1}$ as their product. Under the Kendall τ-metric a PC of length n with minimum distance d can correct up to $\frac{d-1}{2}$ errors caused by charge-constrained errors [9].

F. Parvaresh is supported by IPM in part by grant No. 1401680050.

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The maximum size of a PC of length n and minimum Kendall τ-distance *at least* d is denoted by $P(n, d)$ and a PC attaining this size is said to be optimal. We will show in Proposition 2.1, below, that if d is such that optimal PCs of minimum Kendall τ-distance at least d exist, then there exists an optimal PC with the minimum distance exactly d and therefore one can drop the condition "at least" in the latter definition of optimal codes. Several researchers have presented bounds on $P(n, d)$ (see [1, 2, 9, 11–13]), some of these results are shown in Table 1. It is known that $P(n, 1) = n!$ and $P(n, 2) = \frac{n!}{2}$. Also it is known that if $\frac{2}{3}\binom{n}{2} < d \leq \binom{n}{2}$, then $P(n, d) = 2$ (see [2, Theorem 10]). However, determining $P(n, d)$ turns out to be difficult for $3 \leq d \leq \frac{2}{3}\binom{n}{2}$. In this paper, we study the upper bound of $P(n, 3)$. By sphere packing bound (see [9, Theorems 12 and 13]) $P(n, 3) \leq (n-1)!$. A PC of size $(n-1)!$ and with minimum Kendall τ-distance 3 is called a 1-perfect code. It is proved that if $n > 4$ is a prime number or $4 \leq n \leq 10$, then there is no 1-perfect code in S_n (see [5, Corollary 2.5 and Theorem 2.6] or [2, Corollary 2]).

There are several works using optimization techniques to bound the size of permutation codes under various distance metrics (Hamming, Kendall τ, Ulam) (see [7, 10, 11]). In Section 2, we show that for any non-trivial subgroup of S_n, we can derive an integer programming problem where the optimal value of the objective function gives an upper bound on $P(n, 3)$. In Section 3, by considering the integer programming problem corresponding to the Young subgroups (see Definition 3.1, below) of S_n, we prove the following result:

Theorem 1.1 *For all primes* $p \geq 11$, $P(p, 3) \leq (p-1)! - \lceil \frac{p}{3} \rceil + 2 \leq (p-1)! - 2$.

We then use a software to solve the integer programming problems that are derived from specific choices of the underline subgroup and obtain tighter upper bounds for some small values of n. Finally, we apply a related method from [5] to prove the nonexistent of 1-perfect codes in S_{14}, S_{15}.

2 Preliminaries

A *simple graph* Γ consists of a non-empty set of vertices $V(\Gamma)$ and a possibly empty set of edges $E(\Gamma)$ which is a subset of the set of all 2-element subsets of $V(\Gamma)$. Two vertices σ_1 and σ_2 are called adjacent, denoted by $\sigma_1 - \sigma_2$, if $\{\sigma_1, \sigma_2\} \in E(\Gamma)$. A *subgraph* H of Γ is a simple graph whose vertex set and edge set are subsets of those of Γ. A *path* is a simple graph with the vertex set $\{\sigma_0, \sigma_1, \ldots, \sigma_n\}$ such that $\sigma_j - \sigma_{j+1}$ for $j = 0, \ldots, n-1$. The length of a path is the number of its edges.

Table 1 Some results on the upper bounds of $P(n, 3)$

n	6	7	11	13
Old UB	5!-1[a]	6!-1[a]	10!-1[a]	12!-1[a]
UB	$5! - 4$	$6! - 4$	$10! - 10$	$12! - 12$
n	14	15	17	prime $n \geq 19$
Old UB	13! [9]	14! [9]	16!-1[a]	(n-1)!-1[a]
UB	$13! - 1$	$14! - 1$	$16! - 5$	$(n-1)! - \lceil \frac{n}{3} \rceil + 2$

The superscripts show the references from which the upper bound is taken, where "a" is [2, 5], and gray color shows our main results

By a *graphical code* of minimum distance at least d we mean a subset of vertices of a simple graph such that any two distinct vertices has distance at least d, where the distance of two vertices is defined to be the shortest length of a path between the vertices. Examples of such codes are permutation codes under Kendall τ-metric or Ulam metric, where the vertices of the simple graph are the permutations of length n and two permutations are connected by an edge if and only if their distance under the metric is one. In fact the set of all permutations with the Kendall τ or Ulam metrics can be represented as Cayley graphs (see Definition 2.4, below) and PCs are then subgraphs of the Cayley graph. The methods used in this paper rely on the fact that the permutation set with Kendall τ-metric is a Cayley graph.

Here we observe that if d is such that graphical codes of minimum distance at least d exist, then the ones with the minimum distance exactly d exist.

Proposition 2.1 *Let Γ be any simple graph and $d \geq 1$ an integer. Then*

$$\{|C| \mid C \subseteq V(\Gamma) \text{ and } d_\Gamma(C) = d\} = \{|C| \mid C \subseteq V(\Gamma) \text{ and } d_\Gamma(C) \geq d\},$$

where $d_\Gamma(C) = \min\{d_\Gamma(x, y) \mid x, y \in C \text{ and } x \neq y\}$.

Proof Let C be a graphical code with the minimum distance at least d. Suppose that $\sigma, \tau \in C$ such that $d_\Gamma(C) = d_\Gamma(\sigma, \tau) = d + \ell$ for some non-negative integer ℓ. If $\ell = 0$, we are done; so from now on assume that $\ell > 0$. Let $\sigma - \sigma_1 - \cdots - \sigma_\ell - \cdots - \sigma_{d+\ell-1} - \tau$ be a shortest path in the graph Γ between σ and τ. Consider $\hat{C} = (C \setminus \{\sigma\}) \cup \{\sigma_\ell\}$. We claim that $|C| = |\hat{C}|$ and $d_\Gamma(\hat{C}) = d$, this will complete the proof. If $\sigma_\ell \in C$, then $d(\sigma_\ell, \tau) = d$, which implies $\ell = 0$, a contradiction. It follows that $|C| = |\hat{C}|$. To prove that $d_\Gamma(\hat{C}) = d$, it is enough to show that $d_\Gamma(\delta, \sigma_\ell) \geq d$ for all $\delta \in C \setminus \{\sigma\}$. Since $d_\Gamma(C) = d + \ell$ and by the triangle inequality we have

$$d + \ell \leq d_\Gamma(\delta, \sigma) \leq d_\Gamma(\delta, \sigma_\ell) + d_\Gamma(\sigma_\ell, \sigma) = d_\Gamma(\delta, \sigma_\ell) + \ell.$$

So $d_\Gamma(\delta, \sigma_\ell) \geq d$, as required. □

A PC with Hamming metric is not a graphical code as the Hamming distance between two permutations is never equal to 1 and so we cannot apply Proposition 2.1 for the latter case. We do not know if the conclusion of Proposition of 2.1 is valid for PCs with Hamming metric. We propose the following question.

Question 2.2 *Let d_H be the Hamming metric on S_n and $d \geq 2$ be an arbitrary integer. Is it true that*

$$\{|C| \mid C \subseteq S_n \text{ and } d_H(C) = d\} = \{|C| \mid C \subseteq S_n \text{ and } d_H(C) \geq d\}?,$$

where $d_H(C) = \min\{d_H(x, y) \mid x, y \in C \text{ and } x \neq y\}$.

Definition 2.3 *Let G be a finite group and B, C be two non-empty subsets of G. As usual we denote by BC the set $\{bc \mid b \in B, c \in C\}$, where by $g = bc$ we refer to the group operation. Also, for each $g \in G$ we denote by Bg the set $B\{g\}$. The set B is called inverse closed if $B = B^{-1} := \{b^{-1} \mid b \in B\}$. We also use the notation ξ to denote the identity element of G.*

Let G be a finite group and denote by $\mathbb{C}[G]$ the "complex group algebra" of G. The elements of $\mathbb{C}[G]$ are the formal sums

$$\sum_{g \in G} a_g g, \tag{2.1}$$

where $a_g \in \mathbb{C}$. The complex group algebra is a $\mathbb{C}$-algebra with the following addition, multiplication and scalar product:

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G}(a_g + b_g)\sigma,$$

$$\Big(\sum_{g\in G} a_g g\Big)\Big(\sum_{g\in G} b_g g\Big) = \sum_{g\in G}\Big(\sum_{g=g_1g_2} a_{g_1}b_{g_2}\Big)g,$$

$$\lambda\sum_{g\in G} a_g g = \sum_{g\in G}(\lambda a_g)g,$$

where $\lambda, a_g, b_g \in \mathbb{C}$. If $a_g = 0$ for some g, the term $a_g g$ will be neglected in (2.1) and $\sum_{g\in G} a_g g$ is written as $a_1g_1 + \cdots + a_kg_k$, where $\{g \mid a_g \neq 0\} = \{g_1, \ldots, g_k\}$ is non-empty and otherwise $\sum_{g\in G} a_g g$ is denoted by 0. For a non-empty finite subset Θ of G, we denote by $\widehat{\Theta}$ the element $\sum_{\theta\in\Theta}\theta$ of $\mathbb{C}[G]$.

Definition 2.4 *Let G be a finite group and S be a non-empty inverse closed subset of G not containing the identity element ξ of G. Then the Cayley graph $\Gamma := Cay(G, S)$ is a simple graph with $V(\Gamma) = \{g \mid g \in G\}$ and $E(\Gamma) = \big\{\{g, h\} \,\big|\, g, h \in G, gh^{-1} \in S\big\}$.*

Let G be a finite group and S be a non-empty inverse closed subset of G not containing the identity element ξ of G. Now we have a metric d_Γ on G defined by Γ which is the shortest length of a path between two vertices in $Cay(G, S)$. For example if $G = S_n$ and $S = \{(1, 2), (2, 3), \ldots, (n-1, n)\}$, the metric d_Γ is the Kendall τ-metric on S_n. Also if $G = S_n$ and $S = T \cup T^{-1}$, where $T := \{(a, a+1, \ldots, b) \mid a < b, a, b \in [n]\}$, the metric d_Γ is the Ulam metric on S_n.

Definition 2.5 *For a positive integer r and an element $g \in G$, the ball of radius r in G under the metric d_Γ is denoted by $B_r^\Gamma(g)$ defined by $B_r^\Gamma(g) = \{h \in G \mid d_\Gamma(g, h) \le r\}$.*

Remark 2.6 *Note that $B_r^\Gamma(g) = (S^r \cup \{\xi\})g$, where $S^r := \{s_1 \cdots s_t \mid s_1, \ldots, s_t \in S,\ 1 \le t \le r\}$. Also note that since S is inverse closed, $B_r^\Gamma(g) = S^r g$ for all $r \ge 2$. It follows that $|B_r^\Gamma(g)| = |B_r^\Gamma(\xi)| = |S^r \cup \{\xi\}|$ for all $g \in G$.*

Proposition 2.7 *Let G be a finite group and d_Γ be the metric induced by the graph $Cay(G, S)$. Then a subset C of G is a code with $min\{d_\Gamma(x, y) \mid x, y \in C\} \ge d$ if and only if there exists $Y \subset G$ such that*

$$(\widehat{S^{\lfloor\frac{d-1}{2}\rfloor} \cup \{\xi\}})\widehat{C} = \widehat{G} - \widehat{Y}, \tag{2.2}$$

Proof Let $r := \lfloor\frac{d-1}{2}\rfloor$, $Y = G \setminus \cup_{c\in C} B_r^\Gamma(c)$ and $T := S^r \cup \{\xi\}$. So $G = \cup_{c\in C} B_r^\Gamma(c) \cup Y$. It follows from Remark 2.6 that for each $c \in C$, $B_r^\Gamma(c) = Tc$ and so $\cup_{c\in C} B_r^\Gamma(c) = TC$. Therefore, $\widehat{G} = \widehat{TC} + \widehat{Y}$. On the other hand, for any two distinct elements c, c' in C, $Tc \cap Tc' = \varnothing$ since otherwise $d_\Gamma(c, c') \le d - 1$ that is a contradiction. Hence, $\widehat{TC} = \widehat{T}\widehat{C}$ and this completes the proof. □

Definition 2.8 *Let G be a finite group and d_Γ be the metric induced by $Cay(G, S)$. For a positive integer r, an r-perfect code or a perfect code of radius r of G under the metric d_Γ is a subset C of G such that $G = \cup_{c\in C} B_r^\Gamma(c)$ and $B_r^\Gamma(c) \cap B_r^\Gamma(c') = \varnothing$ for any two distinct $c, c' \in C$.*

Remark 2.9 *By a similar argument as the proof of Proposition 2.7, it can be seen that if C is an r-perfect code, then $(\widehat{S^r \cup \{\xi\}})\widehat{C} = \widehat{G}$. We note that according to Remark 2.6, if C is an r-perfect code then $|C||S^r \cup \{\xi\}| = |G|$.*

Let ρ be any (complex) *representation* of a finite group G of dimension k for some positive integer k, i.e., any group homomorphism from G to the general linear group $\mathrm{GL}_k(\mathbb{C})$ of $k \times k$ invertible matrices over $\mathbb{C}$. Then by the universal property of $\mathbb{C}[G]$, ρ can be extended to an algebra homomorphism $\hat{\rho}$ from $\mathbb{C}[G]$ to the algebra $\mathrm{Mat}_k(\mathbb{C})$ of $k \times k$ matrices over $\mathbb{C}$ such that $g^{\hat{\rho}} = g^{\rho}$ for all $g \in G$. Thus the image of $\widehat{\Theta}$ for any non-empty subset Θ of G under $\hat{\rho}$ is the element $\sum_{\theta\in\Theta} \theta^{\rho}$ of $\mathrm{Mat}_k(\mathbb{C})$. In particular by applying $\hat{\rho}$ on the equality (2.2), we obtain

$$\Big(\sum_{s\in S\cup\{\xi\}} s^{\rho}\Big)\Big(\sum_{c\in C} c^{\rho}\Big) = \sum_{g\in G} g^{\rho} - \sum_{y\in Y} y^{\rho}, \tag{2.3}$$

where the latter equality is between elements of $\mathrm{Mat}_k(\mathbb{C})$.

In the following, we state an important definition that will play a central role in the proof of the main results of this paper.

Definition 2.10
- *Given a group G and a non-empty set Θ, recall that we say G acts on Θ (from the right) if there exists a function $\Theta \times G \to \Theta$ denoted by $(\theta, g) \mapsto \theta^g$ for all $(\theta, g) \in \Theta \times G$ if $(\theta^g)^h = \theta^{gh}$ and $\theta^{\xi} = \theta$ for all $\theta \in \Theta$ and all $g, h \in G$.*
- *For any $\theta \in \Theta$ the set $\mathrm{Stab}_G(\theta) := \{g \in G \mid \theta^g = \theta\}$ is called the stabilizer of θ in G which is a subgroup of G.*
- *If the action is transitive (i.e., for any two elements $\theta_1, \theta_2 \in \Theta$, there exists $g \in G$ such that $\theta_1^g = \theta_2$), all stabilizers are conjugate under the elements of G, more precisely $\mathrm{Stab}_G(\theta_1)^g = \mathrm{Stab}_G(\theta_2)$ whenever $\theta_1^g = \theta_2$, where $\mathrm{Stab}_G(\theta_1)^g = g^{-1}\mathrm{Stab}_G(\theta_1)g$.*
- *Suppose that G acts on Θ and $|\Theta| = k$ is finite. Fix an arbitrary ordering on the elements of Θ so that $\theta_i < \theta_j$ whenever $i < j$ for distinct elements $\theta_i, \theta_j \in \Theta$. Denote by ρ_{Θ}^G the map from G to $\mathrm{GL}_k(\mathbb{Z})$ (the group of all $k \times k$ invertible matrices with integer entries) defined by $g \mapsto P_g$, where P_g is the $k \times k$ matrix whose (i, j) entry is 1 if $\theta_i^g = \theta_j$ and 0 otherwise.*

Remark 2.11 *Note that the definitions of ρ_{Θ}^G depends on the choice of the ordering on Θ, however any two such representations of G are conjugate by a permutation matrix.*

Remark 2.12 *Let H be a subgroup of a finite group G and X be the set of right cosets of H in G, i.e., $X := \{Hg \mid g \in G\}$. Then G acts transitively on X via $(Hg, g_0) \longrightarrow Hgg_0$. It is known that X partitions G, i.e., $G = \cup_{x\in X} x$ and $x \cap x' = \varnothing$ for all distinct elements x and x' of X, and $|X| = |G|/|H|$.*

Lemma 2.13 *Let H be a subgroup of a finite group G and $X = \{Ha_1, \ldots, Ha_m\}$ be the set of right cosets of H in G. If $\mathcal{Y} \subset G$, then by fixing the ordering $Ha_i < Ha_j$ whenever $i < j$, the (i, j) entry of $\sum_{y\in\mathcal{Y}} y^{\rho_X^G}$ is $|\mathcal{Y} \cap a_i{}^{-1}Ha_j|$.*

Proof Clearly, for any $y \in \mathcal{Y}$, the (i, j) entry of $y^{\rho_X^G}$ is 1 if $Ha_i y = Ha_j$ and is 0 otherwise. So the (i, j) entry of $y^{\rho_X^G}$ is 1 if $a_i y a_j{}^{-1} \in H$ and therefore $y \in a_i{}^{-1}Ha_j$. Hence, the (i, j) entry of $\sum_{y\in\mathcal{Y}} y^{\rho_X^G}$ is equal to $|\{y \in \mathcal{Y} \mid y \in a_i{}^{-1}Ha_j\}|$. This completes the proof. □

The following result summarizes the main method used in this paper.

Theorem 2.14 *Let G be a finite group and d_Γ be the metric induced by the graph $Cay(G, S)$. Also, let C be a code in G with $\min\{d_\Gamma(c, c') \mid c \neq c' \in C\} \geq d$. If H is a subgroup of G and X is the set of right cosets of H in G, then the optimal value of the objective function of* *the following integer programming problem gives an upper bound on* $|C|$.

$$\begin{aligned} \textit{Maximize} \quad & \sum_{i=1}^{|X|} x_i, \\ \textit{subject to} \quad & \widehat{T^{\rho_X^G}}(x_1, \ldots, x_{|X|})^t \le |H|\mathbb{1}, \\ & x_i \in \mathbb{Z},\ x_i \ge 0,\ i \in \{1, \ldots, |X|\}, \end{aligned}$$

where $T := S^{\lfloor \frac{d-1}{2} \rfloor} \cup \{\xi\}$, ***1*** *is the column vector of order* $|X| \times 1$ *whose entries are equal to* 1.

Proof Let $r := \lfloor \frac{d-1}{2} \rfloor$. By (2.3), there exists $Y \subset G$ such that

$$\Big(\sum_{s\in T} s^{\rho_X^G}\Big)\Big(\sum_{c\in C} c^{\rho_X^G}\Big) = \sum_{g\in G} g^{\rho_X^G} - \sum_{y\in Y} y^{\rho_X^G}, \tag{2.4}$$

Suppose that $X = \{Ha_1, \ldots, Ha_m\}$. Without loss of generality, we may assume that $a_1 = 1$. We fix the ordering $Ha_i < Ha_j$ whenever $i < j$. By Lemma 2.13, the (i, j) entry of $\sum_{g\in G} g^{\rho_X^G}$ is equal to $|G \cap {a_i}^{-1}Ha_j|$ and since ${a_i}^{-1}Ha_j \subseteq G$, the (i, j) entry of $\sum_{g\in G} g^{\rho_X^G}$ is equal to $|{a_i}^{-1}Ha_j| = |H|$. So if B is a column of $\sum_{g\in G} g^{\rho_X^G}$, then $B = |H|\mathbf{1}$. Let $\mathcal{C}$ be the first column of $\sum_{c\in C} c^{\rho_X^G}$. Then Lemma 2.13 implies that for all $1 \le i \le |X|$, i-th row of $\mathcal{C}$, denoted by c_i, is equal to $|C \cap Ha_i|$. Since $C = C \cap G = \cup_{i=1}^{|X|}(C \cap Ha_i)$ and $(C \cap Ha_i) \cap (C \cap Ha_j) = \varnothing$ for all $i \ne j$, $\sum_{i=1}^{|X|} c_i = |C|$. We note that by Lemma 2.13, all entries of matrix $\widehat{F^{\rho_X^G}}$, $F \in \{C, G, Y, T\}$, are integer and non-negative. Therefore $\mathcal{C}$ is an integer solution for the following system of inequalities

$$\widehat{T^{\rho_X^G}}(x_1, \ldots, x_{|X|})^t \le |H|\mathbb{1}$$

such that $\sum_{i=1}^{|X|} c_i = |C|$ and this completes the proof. □

3 Results

Let $G = S_n$ and $S = \{(i, i+1) \mid 1 \le i \le n-1\}$. Then the metric induced by $Cay(G, S)$ on S_n is the Kendall τ-metric. In this section, by using the results in Section 2, we improve the upper bound of $P(n, 3)$ when $n \in \{6, 14, 15\}$ or $n \ge 7$ is a prime number. We note that for two permutations σ and λ of S_n, their multiplication $\lambda \cdot \sigma$ is defined as the composition of σ on λ, namely $\lambda \cdot \sigma(i) = \sigma(\lambda(i))$ for all $i \in [n]$.

In order to apply Theorem 2.14, we need to fix the subgroup H. Clearly, different choices for H will lead to different results. Throughout this paper, H will be chosen from the collection of all Young subgroups, which are well studied subgroups of S_n (see [8]). The definition of Young subgroup is given next.

Definition 3.1 *By a number partition* λ *of* n *(with the length* m*) we mean an* m*-tuple* $(\lambda_1, \ldots, \lambda_m)$ *of positive integers such that* $\lambda_1 \ge \cdots \ge \lambda_m$ *and* $n = \sum_{i=1}^m \lambda_i$. *If* λ *and* μ *are two partitions of* n, *we say that* λ *dominates* μ, *and write* $\lambda \trianglelefteq \mu$, *provided that* $\sum_{i=1}^j \lambda_i \ge \sum_{i=1}^j \mu_i$ *for all* j. *Let* λ *be a partition of* n *and* $\Delta := (\Delta_1, \ldots, \Delta_m)$ *be an* m*-tuple of non-empty subsets of* $[n]$ *consisting of a set partition for* $[n]$ *with* $|\Delta_i| = \lambda_i$ *for* *all $i = 1, \ldots, m$. We associate a Young subgroup S_Δ of S_n by taking $S_\Delta = S_{\Delta_1} \times \cdots \times S_{\Delta_m}$, where S_{Δ_i} is the symmetric group on the set Δ_i for all $i = 1, \ldots, m$.*

Remark 3.2 *Let λ be a partition of n and Δ, Δ' be two m-tuples of non-empty subsets of $[n]$ consisting of a set partition for $[n]$ with $|\Delta_i| = |\Delta'_i| = \lambda_i$ for all $i = 1, \ldots, m$. It is known that the representations $\rho_X^{S_n}$ and $\rho_{X'}^{S_n}$, where X and X' are the set of right cosets of the Young subgroups S_Δ and $S_{\Delta'}$ in S_n, respectively, are equivalent (i.e., an invertible matrix U exists such that $U^{-1}\rho_X^{S_n}(\sigma)U = \rho_{X'}^{S_n}(\sigma)$ for all $\sigma \in S_n$). Hence, we use the m-tuples of non-empty subsets of $[n]$, $[\{1, \ldots, \lambda_1\}, \{\lambda_1+1, \ldots, \lambda_1+\lambda_2\}, \ldots, \{n-\lambda_m+1, \ldots, n\}]$ for considering the Young subgroup corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, as we are studying these representations up to equivalence.*

For example, if $n = 7$ and $\lambda = (3, 2, 2)$, then the Young subgroup corresponding to the partition λ is the subgroup $H = \{\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \mid \sigma_1 \in S_3, \sigma_2 \in S_{\{4,5\}}, \sigma_3 \in S_{\{6,7\}}\}$.

Lemma 3.3 *Let H be a Young subgroup of S_n corresponding to the partition $\lambda := (n-1, 1)$ and X be the set of right cosets of H in S_n. If $S = \{(i, i+1) \mid 1 \le i \le n-1\}$ and $T := S \cup \{\xi\}$, then $\widehat{T^{\rho_X^{S_n}}}$ is a conjugate by a permutation matrix of the following matrix*

$$\begin{pmatrix} n-1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & n-2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & n-2 & 1 & 0 & 0 \\ \vdots & \ldots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & n-2 & 1 \\ 0 & 0 & \ldots & 0 & 1 & n-1 \end{pmatrix}. \tag{3.1}$$

Proof Without loss of generality we may assume that λ is the partition $\{\{1\}, \{2, \ldots, n\}\}$ of n and therefore $H = \mathrm{Stab}_{S_n}(1)$. Clearly, for each $i \in [n]$, if $\sigma \in H(1, i)$, then $\sigma(1) = i$ and so $H(1, i) \cap H(1, j) = \varnothing$ for all $i \ne j$. So we can let $X = \{H(1, i) \mid 1 \le i \le n\}$, where we are using the convention $H(1, 1) := H$. Fix the ordering of X such that $H(1, i) < H(1, j)$ if $i < j$. By Lemma 2.13, the (i, j) entry of $\widehat{T^{\rho_X^{S_n}}}$ is equal to $|T \cap (1, i)H(1, j)|$. If $i = j$, then Definition 2.10 implies $(1, i)H(1, i) = \mathrm{Stab}_{S_n}(i)$ and hence $T \cap (1, i)H(1, i) = T \setminus \{(i-1, i), (i, i+1)\}$ if $2 \le i \le n-1$, $T \cap (1, n)H(1, n) = T \setminus \{(n, n-1)\}$ and $T \cap H = T \setminus \{(1, 2)\}$. Now suppose that $i \ne j$. Clearly $(1, i) \cdot (i, j) \cdot (1, j) = (i, j)$. Let $h \in H$. Then $\sigma := (1, i) \cdot h \cdot (1, j) = \pi(1, j, i)$, where $\pi = (1, i) \cdot h \cdot (1, i) \in \mathrm{Stab}_{S_n}(i)$. Since $\pi(i) = i$, $\sigma(j) = i$ and therefore σ is an transposition if and only if $h = (i, j)$. Hence, if $j = i+1$ and $i-1$, then $T \cap (1, i)H(1, j)$ is equal to $\{(i, i+1)\}$ and $\{(i-1, i)\}$, respectively, and otherwise $T \cap (1, i)H(1, j) = \varnothing$. This completes the proof. □

Theorem 3.4 *Let $p \ge 7$ be a prime number and consider the $p \times p$ matrix*

$$M = \begin{pmatrix} p-1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & p-2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & p-2 & 1 & 0 & 0 \\ \vdots & \ldots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & p-2 & 1 \\ 0 & 0 & \ldots & 0 & 1 & p-1 \end{pmatrix}.$$

Consider the system of inequalities $M(x_1, \ldots, x_p)^t \le (p-1)!\mathbf{1}$ with $(x_1, \ldots, x_p)^t \ge \mathbf{0}$ and x_i are integers. Let $x_{\max} := \max\{x_i \mid i = 1, \ldots, p\}$. Then

(1) $|\{i \in [p] \mid x_i \leq \frac{(p-1)!}{p}\}| \geq \lceil \frac{p}{3} \rceil$.
(2) *If* $\sum_{i=1}^{p} x_i = (p-1)! - k$, *then* $|\{i \mid x_i = x_{\max}\}| \geq p - k - 2$.
(3) $\sum_{i=1}^{p} x_i \leq (p-1)! - \lceil \frac{p}{3} \rceil + 2$

Proof Let $\mathcal{A} := \{i \in [p] \mid x_i \leq \frac{(p-1)!}{p}\}$ and $\mathcal{B} := \{i \mid x_i = x_{\max}\}$. Consider the partition $\{\{1,2\}, \{3,4,5\}, \{6,7,8\}, \ldots, \{p-2, p-1, p\}\}$ of $[p]$ if $p \equiv 2 \mod 3$ and the partition $\{\{1,2\}, \{3,4,5\}, \{6,7,8\}, \ldots, \{p-4, p-3, p-2\}, \{p-1, p\}\}$ if $p \equiv 1 \mod 3$. Each member of partitions corresponds to an obvious inequality, e.g. $\{1,2\}$ and $\{p-2, p-1, p\}$ are respectively corresponding to $(p-1)x_1 + x_2 \leq (p-1)!$ and $x_{p-2} + (p-2)x_{p-1} + x_p \leq (p-1)!$. Each inequality corresponding to a member P of the partitions forces $x_i \leq (p-1)!/p$ for some $i \in P$, where $x_i = \min\{x_j \mid j \in P\}$. Since the size of both partitions is $\lceil \frac{p}{3} \rceil$, we have that $|\mathcal{A}| \geq \lceil \frac{p}{3} \rceil$ and so the first part is proved.

It follows from $M(x_1, \ldots, x_p)^t \leq (p-1)!\mathbf{1}$ and $(x_1, \ldots, x_p)^t \geq \mathbf{0}$ that $0 \leq \sum_{i=1}^{p} M_i \mathbf{x} = p(\sum_{i=1}^{p} x_i) \leq p!$, where M_i is i-th row of M and so $0 \leq \sum_{i=1}^{p} x_i \leq (p-1)!$. Let $\ell \in [p]$ be such that $x_\ell = x_{\max}$. Thus $\sum_{i=1, i \neq \ell-1, \ell+1}^{p} (x_\ell - x_i) = x_{\ell-1} + (p-2)x_\ell + x_{\ell+1} - \sum_{i=1}^{p} x_i \leq (p-1)! - ((p-1)! - k)$. Thus $\sum_{i=1, i \neq \ell-1, \ell+1}^{p} (x_\ell - x_i) \in \{0, 1, \ldots, k\}$. It follows that $|\{i \mid x_i < x_{\max}\}| \leq k+2$ and so $|\mathcal{B}| \geq p - k - 2$ and the second part is proved.

Let $\sum_{i=1}^{p} x_i = (p-1)! - k$ and suppose, for a contradiction, that $k < \lceil \frac{p}{3} \rceil - 2$. So $|\mathcal{B}| \geq p - \lceil \frac{p}{3} \rceil + 1$ and therefore

$$|\mathcal{A} \cap \mathcal{B}| \geq |\mathcal{A}| + |\mathcal{B}| - p \geq \lceil \frac{p}{3} \rceil + p - \lceil \frac{p}{3} \rceil + 1 - p \geq 1.$$

Hence $\mathcal{A} \cap \mathcal{B} \neq \varnothing$ and $x_{\max} \leq (p-1)!/p$. Since p is prime, by Wilson theorem [4, P. 27] $(p-1)! \equiv -1 \mod p$. Since $x_{\max}$ is integer, we have that $x_i \leq \frac{(p-1)!+1}{p} - 1$ for all $i \in [p]$. Therefore

$$\sum_{i=1}^{p} x_i = (p-1)! - k \leq p(\frac{(p-1)!+1}{p} - 1) = (p-1)! + 1 - p$$

and so

$$p \leq k + 1 < \lceil \frac{p}{3} \rceil - 1,$$

which is a contradiction. So we must have $k \geq \lceil \frac{p}{3} \rceil - 2$. This completes the proof. □

In the following we will prove Theorem 1.1.

Theorem 3.5 *For all primes* $p \geq 11$, $P(p,3) \leq (p-1)! - \lceil \frac{p}{3} \rceil + 2 \leq (p-1)! - 2$.

Proof Let C be a code in S_p with minimum Kendall τ-distance 3. Let H be the Young subgroup of S_p corresponding to the partition $\lambda := (p-1, 1)$ and X be the set of right cosets of H in S_p. If $S = \{(i, i+1) \mid 1 \leq i \leq p-1\}$ and $T := S \cup \{\xi\}$, then by Lemma 3.3, $\widehat{T^{\rho_X^{S_n}}}$ is a conjugate by a permutation matrix of the matrix M in Theorem 3.4. Now Theorem 2.14 implies that the optimal value of the objective function of the following integer programming problem gives an upper bound on $|C|$

$$\begin{aligned} \text{Maximize} \quad & \sum_{i=1}^{p} x_i, \\ \text{subject to} \quad & M(x_1, \ldots, x_p)^t \leq |H|\mathbf{1} = (n-1)!\mathbf{1}, \\ & x_i \in \mathbb{Z}, \ x_i \geq 0, \ i \in \{1, \ldots, p\}, \end{aligned}$$

where **1** is a column vector of order $p \times 1$ whose entries are equal to 1. Therefore, the result follows from Theorem 3.4. This completes the proof.

Theorem 3.6 *If n is equal to 6, 7, 11, 13 and 17, then $P(n, 3)$ is less than or equal to 116, 716, $10! - 10$, $12! - 12$ and $16! - 5$, respectively.*

Proof Let $S := \{(i, i+1) \mid 1 \leq i \leq n-1\}$. In view of Theorem 2.14, we have used CPLEX software [3] and GAP software [6] to determine the upper bound for $P(n, 3)$ obtained from solving the integer programming problem corresponding to the subgroup H of S_n, where H is the Young subgroup corresponding to the partition (2, 2, 2), when $n = 6$, (5, 1, 1), when $n = 7$, (9, 2), when $n = 11$, (11, 2), when $n = 13$ and (16, 1), when $n = 17$. For each of the above subgroups, using GAP software [6], first, we determined the matrix $\widehat{(T)^{\rho_X^{S_n}}}$, where X is the set of right cosets of H in S_n and $T := S \cup \{\xi\}$, then using CPLEX software [3], we solved the integer programming problem corresponding to the subgroup H.

To prove the non-existence of 1-perfect codes in S_{14} and S_{15}, we are using techniques in [5] which is stated in the following proposition.

Proposition 3.7 *[5, Theorem 2.2] Let $S = \{(i, i+1) \mid 1 \leq i \leq n-1\}$ and $T := S \cup \{\xi\}$. If S_n contains a subgroup H such that $n \nmid |H|$ and $\widehat{(T)^{\rho_X^{S_n}}}$ is invertible, where X is the set of right cosets of H in S_n, then S_n contains no 1-perfect codes.*

Theorem 3.8 *There are no 1-perfect codes under the Kendall τ-metric in S_n when $n \in \{14, 15\}$.*

Proof Let $S = \{(i, i+1) \mid 1 \leq i \leq n-1\}$ and $T := S \cup \{\xi\}$. By Proposition 3.7, to prove the non-existence of 1-perfect codes under the Kendall τ-metric in S_n, we need to show the existence of a subgroup H of $S_n, n \in \{14, 15\}$, with following two properties: (1) $n \nmid |H|$; (2) the matrix $\widehat{(T)^{\rho_X^{S_n}}}$ is invertible. Since $\widehat{(T)^{\rho_X^{S_n}}}$ is a matrix of dimension $n!/|H|$, by choosing H with a larger size, the dimension of the matrix $\widehat{(T)^{\rho_X^{S_n}}}$ decreases. In the case $n = 14$, we consider the Young subgroup H corresponding to the partition (6, 6, 2). It is clear that $14 \nmid |H| = 6!6!2!$. Also, by a computer check the matrix $\widehat{(T)^{\rho_X^{S_{14}}}}$ which is a matrix of dimension 84084 is invertible and so there are no 1-perfect codes under the Kendall τ-metric in S_{14}. In the case $n = 15$, the largest Young subgroup H of S_{15} which satisfies the condition (1) is the Young subgroup corresponding to the partition $\lambda := (4, 4, 4, 3)$. In this case the matrix $\widehat{(T)^{\rho_X^{S_{15}}}}$ is of dimension 1051050 that the software was unable to check its invertibility. Therefore, we use [8, Corollary 2.2.22] to check its invertibility. By [8, Corollary 2.2.22], if for all partitions μ of n which $\mu \trianglelefteq \lambda$, $\widehat{T^{\rho_\mu}}$ are invertible, where ρ_μ is the irreducible representation of S_{15} corresponding to μ, then $\widehat{T^{\rho_X^{S_{15}}}}$ is invertible. There exist 54 partitions of 15 which dominates the partition λ. By computer check, for each partition μ of these 54 partition the matrix $\widehat{T^{\rho_\mu}}$ is invertible (Table 2 shows the dimension and the eigenvalue with smallest absolute value of theses martices) and so $\widehat{(T)^{\rho_X^{S_{15}}}}$ is invertible and this completes the proof. □

Conjecture 3.9 *If H is the Young subgroup corresponding to the partition $(p-1, p-1, 2)$ of S_{2p}, where $p \geq 3$ is a prime number, and X is the set of right cosets of H in S_{2p}, then*

Table 2 The dimension and the eigenvalue with smallest absolute value of the matrix $\widehat{T^{\rho_\mu}}$ for all partitions μ of 15 which dominates the partition (4,4,4,3)

item	Partition	Dimension	Eigenvalue
1	(15)	1	1.5×10
2	(14, 1)	14	1.104×10
3	(13, 2)	90	7.232
4	(13, 1, 1)	91	7.217
5	(12, 3)	350	3.686
6	(12, 2, 1)	715	3.619
7	(11, 4)	910	5.467×10^{-1}
8	(8, 7)	1430	-8.095×10^{-3}
9	(10, 5)	1638	-2.611×10^{-3}
10	(11, 2, 2)	1925	3.149×10^{-1}
11	(9, 6)	2002	-4.503×10^{-3}
12	(11, 3, 1)	2835	3.745×10^{-1}
13	(7, 7, 1)	5005	1.035×10^{-2}
14	(5, 5, 5)	6006	-1.497×10^{-3}
15	(10, 4, 1)	7007	-7.028×10^{-3}
16	(10, 3, 2)	9100	1.12×10^{-2}
17	(9, 5, 1)	11375	2.99×10^{-3}
18	(8, 6, 1)	11583	-4.224×10^{-3}
19	(9, 3, 3)	12740	-4.444×10^{-3}
20	(9, 2, 2, 2)	13650	-1.825×10^{-4}
21	(6, 6, 3)	21450	-1.728×10^{-6}
22	(9, 4, 2)	22113	9.626×10^{-5}
23	(4, 4, 4, 3)	24024	-8.294×10^{-4}
24	(7, 6, 2)	25025	-2.424×10^{-4}
25	(7, 4, 4)	25025	1.108×10^{-3}
26	(6, 5, 4)	30030	1.033×10^{-5}
27	(8, 5, 2)	32032	8.217×10^{-4}
28	(8, 4, 3)	35035	-2.925×10^{-4}
29	(7, 5, 3)	45045	3.142×10^{-5}
30	(6, 3, 3, 3)	50050	3.128×10^{-5}
31	(8, 3, 2, 2)	58968	3.477×10^{-4}
32	(5, 4, 3, 3)	75075	-1.733×10^{-4}
33	(5, 4, 4, 2)	81081	-5×10^{-5}
34	(7, 3, 3, 2)	90090	2.1×10^{-5}
35	(5, 5, 3, 2)	96525	-5.987×10^{-5}
36	(6, 5, 2, 2)	100100	2.946×10^{-5}
37	(7, 4, 2, 2)	112112	-6.787×10^{-5}
38	(6, 4, 3, 2)	175175	-3.594×10^{-5}
39	(12, 1, 1, 1)	364	3.599
40	(11, 2, 1, 1)	2925	2.852×10^{-1}

Table 2 continued

item	Partition	Dimension	Eigenvalue
41	(10, 2, 2, 1)	9450	-1.485×10^{-4}
42	(10, 3, 1, 1)	11088	5.085×10^{-3}
43	(9, 4, 1, 1)	25025	-1.767×10^{-4}
44	(7, 6, 1, 1)	27027	3.757×10^{-4}
45	(8, 5, 1, 1)	35100	-7.440×10^{-5}
46	(9, 3, 2, 1)	42042	6.633×10^{-4}
47	(6, 6, 2, 1)	50050	-1.680×10^{-5}
48	(5, 5, 4, 1)	54054	1.934×10^{-4}
49	(8, 3, 3, 1)	57330	9.513×10^{-5}
50	(6, 4, 4, 1)	80080	-2.972×10^{-5}
51	(8, 4, 2, 1)	91000	-2.590×10^{-5}
52	(7, 5, 2, 1)	108108	-3.672×10^{-5}
53	(6, 5, 3, 1)	128700	-1.920×10^{-5}
54	(7, 4, 3, 1)	135135	-2.627×10^{-6}

$\widehat{(S \cup \{\xi\})^{\rho_X}}^{S_{2p}}$ *is invertible. In particular, there is no* 1*-perfect permutation code of length* $2p$ *with respect to the Kendall* τ*-metric.*

We note that by computer checking Conjecture 3.9 holds valid for $p \in \{3, 5, 7\}$.

4 Conclusion

Due to the applications of PCs under the Kendall τ-metric in flash memories, they have attracted the attention of many researchers. In this paper, we consider the upper bound of the size of the largest PC with minimum Kendall τ-distance 3. Using group theory, we formulate an integer programming problem depending on the choice of a non-trivial subgroup of S_n, where the optimal value of the objective function gives an upper bound on $P(n, 3)$. After that, by solving the integer programming problem corresponding to some subgroups of S_n, when $n \geq 7$ is a prime number or $n \in \{6, 14, 15\}$, we improve the upper bound on $P(n, 3)$.

Author Contributions All authors read and approved the final manuscript.

Funding This research is supported by the Deputy of Research and Technology of University of Isfahan under a grant given to the research group CSG (Code-Scheme-Group). F. Parvaresh is also supported by IPM in part by grant No. 1401680050. These funding sources had no role in the design of this study and will not have any role during its execution, analyses, interpretation of the data, or decision to submit results.

Availability of supporting data All data generated or analysed during this study are included in this published article.

Declarations

Ethical Approval and Consent to participate Not applicable. The current manuscript does not report on or involve the use of any animal or human data or tissue.

Competing Interests The authors declare that they have no competing interests.

Consent for publication Not applicable. The current manuscript does not contain data from any individual person.

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