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ON SHARP CHARACTERS OF TYPE $\{-1,0,2\}$

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Abstract. For a complex character χ of a finite group G, it is known that the product $\mathrm{sh}(\chi)=\prod_{l\in L(\chi)}(\chi(1)-l)$ is a multiple of $|G|$, where $L(\chi)$ is the image of χ on $G-\{1\}$. The character χ is said to be a sharp character of type L if $L=L(\chi)$ and $\mathrm{sh}(\chi)=|G|$. If the principal character of G is not an irreducible constituent of χ, then the character χ is called normalized. It is proposed as a problem by P. J. Cameron and M. Kiyota, to find finite groups G with normalized sharp characters of type $\{-1,0,2\}$. Here we prove that such a group with nontrivial center is isomorphic to the dihedral group of order 12.

Keywords: sharp character; sharp pair; finite group

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1. Introduction

Let G be a finite group, χ be a (complex) character of G, and $L(\chi)$ be the image of χ on $G-\{1\}$. Put $\mathrm{sh}(\chi)=\prod_{l\in L(\chi)}(\chi(1)-l)$. It is known that for any complex character χ of a finite group G, the order of G divides $\mathrm{sh}(\chi)$, see [3]. The pair (G,χ) (or briefly, the character χ) is called *sharp* of type L if $L=L(\chi)$ and $\mathrm{sh}(\chi)=|G|$. It is obvious that χ is faithful whenever (G,χ) is sharp. The pair (G,χ) (or briefly, the character χ) is said to be normalized if $(\chi,1_G)_G=0$, where 1_G is the principal character of G and the product $(\chi,\theta)_G$ of two characters χ and θ of G is defined as:

$$(\chi,\theta)_G:=\frac{1}{|G|}\sum_{g\in G}\chi(g)\theta(g^{-1}).$$

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In [5], Cameron and Kiyota posed the problem of classifying normalized sharp pairs (G,χ) of type L for a given set L of algebraic integers. The case that L contains at least an irrational value has been settled by Alvis and Nozawa, see [2]. However, there are few results for the case that L contains only rational integers, see [4], [5], [9], [10].

By [5], Propositions 1.2 and 1.3, if (G,χ) is sharp of type $\{l\}$ and normalized, then $l=-1$ and $\chi=\varrho_G-1_G$, where ϱ_G is the regular character of G and if (G,χ) is normalized and sharp of type $L=\{l_1,l_2\}$, where l_1 and l_2 are distinct rational integers, then $(\chi,\chi)_G=1-l_1l_2$ and $l_1<0\leqslant l_2$. This implies that $(\chi,\chi)_G=1$ if and only if (G,χ) is of type $\{l,0\}$, where $l<0$; $(\chi,\chi)_G=2$ if and only if (G,χ) is of type $\{-1,1\}$; and $(\chi,\chi)_G=3$ if and only if (G,χ) is of type $\{-1,2\}$ or $\{-2,1\}$. For the first case, some properties of G and χ have been stated in [5], [11], and the sharp pairs of the second case have been given in [4]. Also the last case was settled for groups with nontrivial centers in [10], which was generalized to the case $(\chi,\chi)_G=p$, where p is an odd prime and $L(\chi)=\{l,l+p\}$, for $l=-1$ or $1-p$, see [12]. Furthermore, the normalized sharp pairs (G,χ) of type $L=\{\varepsilon,-3\varepsilon\}$, where $\varepsilon=\pm1$ and the center $Z(G)$ of G is nontrivial, have been studied in [1]. In Problem 7.5 of [5], it is proposed to find finite groups G having a normalized sharp character χ of type $L=\{-1,0,2\}$. In this paper, we study these groups G under the additional hypothesis $Z(G)>1$, and we prove the following theorem:

Main Theorem. *Suppose that (G,χ) is normalized and sharp of type $L=\{-1,0,2\}$ and $Z(G)>1$. Then G is isomorphic to the dihedral group D_{12} of order 12.*

To prove our main theorem, we show that χ is the sum of two distinct real valued irreducible characters of G, and $\chi(1)$ is odd.

For groups with trivial center in Problem 7.5 of [5], we just consider simple groups having a normalized sharp irreducible character. In Lemma 2.1, simple groups with normalized sharp irreducible character of type $L=\{-1,0,2\}$ are characterized, using the fact that there exist exactly three simple groups having a faithful irreducible character χ with exactly four distinct values $\chi(1)$, -1, 0, 2, see [10].

2. Main results

Throughout this paper, G is a finite group having a normalized sharp character χ of type $L=\{-1,0,2\}$. Set $n:=\chi(1)$. Since χ is sharp, $|G|=n(n+1)(n-2)=n^3-n^2-2n$ and $n\geqslant3$.

Lemma 2.1. *Suppose that G is a simple group. If χ is irreducible, then n is even and G is isomorphic to either* $\mathrm{PSL}(2,7)$ *or* A_7.

Proof. By [10], proof of Claim B6, there exist exactly three simple groups having a faithful irreducible character χ which takes exactly four distinct values $\chi(1)$, -1, 0, 2. Those are $\mathrm{PSL}(2,7)$ with $\chi(1)=6$, alternating group A_7 with $\chi(1)=14$ and $\mathrm{PSL}(3,3)$ with $\chi(1)=26$. The character χ is sharp of type $\{-1,0,2\}$, for groups $\mathrm{PSL}(2,7)$ and A_7. $\square$

Lemma 2.2. *Let $g\in G$ and $o(g)=2$.*

(1) *If n is even, then $\chi(g)\in\{0,2\}$.*

(1) *If n is odd, then $\chi(g)=-1$.*

Proof. By [4], proof of Proposition 3, if θ is a rational valued character of a finite group G, $y\in G$ and s is a prime, then $\theta(y^s)\equiv\theta(y) \bmod s$.

(1) Since $o(g)=2$, we have $\chi(g)\equiv\chi(1)=n \bmod 2$. Therefore, $\chi(g)\in\{0,2\}$.

(2) Note that $\chi(g)\equiv\chi(1)=n \bmod 2$. Now since n is odd, it follows that $\chi(g)=-1$. $\square$

Lemma 2.3. *If $Z(G)>1$, then χ is a sum of two distinct real valued irreducible characters of G.*

Proof. Note that by [5], Proposition 1.3 (ii), $(\chi,\chi)_G\leqslant 2$. First assume that χ is an irreducible character of G. Since χ is faithful, it follows from [7], Lemma 2.27 (f) that $Z(G)=Z(\chi):=\{g\in G\colon |\chi(g)|=\chi(1)\}$. Therefore, $\chi(g)=-n$ for every nontrivial element $g\in Z(G)$, which implies that $n=1$. This is a contradiction and so $(\chi,\chi)_G=2$. Hence, $\chi=\chi_1+\chi_2$, where χ_1 and χ_2 are distinct irreducible characters of G.

Since χ is rational valued, it follows that $\chi_1+\chi_2=\overline{\chi_1}+\overline{\chi_2}$. As complex conjugate of an irreducible character is also irreducible and irreducible characters are linearly independent, it follows that either $\overline{\chi_1}=\chi_2$ or both χ_1 and χ_2 are real valued. First suppose that $\chi=\chi_1+\overline{\chi_1}$ is the sum of two complex conjugate irreducible characters of G. We show that χ_1 is faithful. Let $g\in\ker(\chi_1)$. Therefore, $\chi(g)=\chi_1(g)+\overline{\chi_1(g)}=\chi_1(1)+\chi_1(1)=\chi(1)$, and so $g=1$ since χ is faithful. Hence, χ_1 is faithful. Now by [7], Theorem 2.32 (a), $Z(G)$ is cyclic. Suppose that $Z(G)=\langle z\rangle$ and $o(z)=r>1$. As $\chi=\chi_1+\overline{\chi_1}$, by [7], Lemma 2.27 (c), we have $\chi(z)=\chi_1(1)(\xi+\overline{\xi})$, where ξ is a primitive rth root of unity since χ is faithful. As $\chi(z)$ is rational, it follows that $r\in\{2,3,4,6\}$. If $r=2$, then $\chi(z)=-2\chi_1(1)\in\{-1,0,2\}$, which is impossible. If $r=3$, then $\xi+\overline{\xi}=2\cos(\frac{2}{3}\pi)=-1$ and $\chi(z)=-\chi_1(1)\in\{-1,0,2\}$, which contradicts $n\geqslant 3$. Now suppose $r=6$. Then $\xi+\overline{\xi}=2\cos(\frac{1}{3}\pi)=1$ and $\chi(z)=\chi_1(1)\in\{-1,0,2\}$.

Therefore, $\chi_1(1) = 2$, $n = 4$ and $|G| = 40$. It is easy to check all groups of order 40 by GAP (see [6]) to see none of them have the requested property. Hence, $r = 4$ and $Z(G) = \langle z\rangle \cong C_4$. Then $\chi(z^2) = \chi_1(1)(\eta+\overline{\eta})$, where η is the primitive square root of unity. Therefore, by Lemma 2.2, $\chi(z^2) = -2\chi_1(1) \in \{0, 2\}$, which is a contradiction. Hence, both χ_1 and χ_2 are real valued and this completes the proof. □

In the sequel of the paper, we assume that χ is the sum of two distinct real valued irreducible characters χ_1 and χ_2 of G.

Lemma 2.4.
(1) $Z(G) = \bigcap\limits_{i=1}^{2} Z(\chi_i)$.
(2) *$Z(G)$ is the direct product of at most two cyclic subgroups.*

Proof. (1) Since χ is faithful, the intersection of kernels of irreducible constitutes of χ is trivial. Now (1) follows from the proof of [7], Corollary 2.28.

(2) Since $\bigcap\limits_{i=1}^{2}\ker(\chi_i) = 1$, it follows that G can be embedded into $\prod\limits_{i=1}^{2} G/\ker(\chi_i)$ and so $Z(G)$ is isomorphic to a subgroup of $\prod\limits_{i=1}^{2} Z(G/\ker(\chi_i))$. By Lemma 2.27 (f) of [7], $Z(G) \hookrightarrow \prod\limits_{i=1}^{2} Z(\chi_i)/\ker(\chi_i)$. Now Lemma 2.27 (d) of [7] completes the proof. □

Lemma 2.5.
(1) *$Z(G)$ is an elementary abelian 2-group of order at most 4.*
(2) *If z is a nontrivial element of $Z(G)$, then*

$$(\chi_1(z), \chi_2(z)) \in \{(\chi_1(1), -\chi_2(1)), (-\chi_1(1), \chi_2(1))\}.$$

Proof. (1) By Lemma 2.4 (1), $Z(G) = Z(\chi_1) \cap Z(\chi_2)$. Since both χ_1 and χ_2 are real valued, it follows from [7], Lemma 2.27 (c) that $\chi_i(z) = \pm\chi_i(1)$ and so $\chi_i(z^2) = \chi_i(1)$ for all $z \in Z(G)$ and $i \in \{1, 2\}$. Thus, $\chi(z^2) = \chi_1(z^2)+\chi_2(z^2) = \chi(1)$ and so $z^2 = 1$ since χ is faithful. Now Lemma 2.4 (2) completes the proof.

(2) By the proof of part (1) we have $\chi_i(z) = \pm\chi_i(1)$ for a nontrivial element $z \in Z(G)$ and $i = 1, 2$. Since $\chi(z) = \chi_1(z) + \chi_2(z) \geqslant -1$ and $\chi(z) \neq \chi(1) = \chi_1(1) + \chi_2(1)$ (χ is faithful), the result follows. □

Lemma 2.6. $|Z(G)| \leqslant 2$.

Proof. We first claim that there exists at most one element $z \in Z(G)$ such that $(\chi_1(z), \chi_2(z)) = (\chi_1(1), -\chi_2(1))$. Suppose that there exist elements $z_1, z_2 \in Z(G)$ such that

$$(\chi_1(z_1), \chi_2(z_1)) = (\chi_1(z_2), \chi_2(z_2)) = (\chi_1(1), -\chi_2(1)).$$

Now we have $\chi(z_1z_2) = \chi_1(z_1z_2) + \chi_2(z_1z_2)$. By [7], Lemma 2.27 (c), there exists linear character λ_1 of $Z(\chi_1)$ such that

$$\chi_1(z_1z_2) = \chi_1(1)\lambda_1(z_1z_2) = \chi_1(1)\lambda_1(z_1)\lambda_1(z_2) = \chi_1(1)\lambda_1(z_2) = \chi_1(z_2) = \chi_1(1).$$

Similarly, we have $\chi_2(z_1z_2) = \chi_2(1)$. Therefore, $\chi(z_1z_2) = \chi(1)$ and so $z_1z_2 = 1$. Hence, $z_1 = z_2$ by Lemma 2.5 (1), as we claimed.

By a similar argument one can prove that there exists at most one element $z' \in Z(G)$ such that $(\chi_1(z'), \chi_2(z')) = (-\chi_1(1), \chi_2(1))$.

Now Lemma 2.5 (2) implies that $|Z(G)| \leqslant 3$ and so by Lemma 2.5 (1) we have $|Z(G)| \leqslant 2$. □

Remark 2.7. In view of Lemmas 2.5 (2) and 2.6, whenever $Z(G) \neq 1$, we shall assume without loss of generality that there exists a (unique) nontrivial element $z \in Z(G)$ such that $\chi_1(z) = \chi_1(1)$, $\chi_2(z) = -\chi_2(1)$.

Lemma 2.8. *Suppose that n is even and $Z(G) > 1$. Then*
(1) $\chi_1(g) \in \{0, \pm1, 2\}$ *and* $\chi_2(g) \in \{0, \pm1\}$ *for all* $g \in G \setminus Z(G)$.
(2) $\ker(\chi_1) = Z(G)$.
(3) $\ker(\chi_2) = 1$.

P r o o f. (1) By Remark 2.7 assume that there exists a nontrivial element $z \in Z(G)$ such that $\chi_1(z) = \chi_1(1)$, $\chi_2(z) = -\chi_2(1)$. Note that if $\mathcal{X}_i$ is a representation corresponding to χ_i for $i = 1, 2$, then $\mathcal{X}_1(z) = I_{\chi_1(1)}$ and $\mathcal{X}_2(z) = -I_{\chi_2(1)}$ by [7], Lemma 2.27. Therefore, $\chi(gz) = \chi_1(g) - \chi_2(g)$ for all $g \in G$. Thus, $\chi(g) + \chi(gz) = 2\chi_1(g)$ and $\chi(g) - \chi(gz) = 2\chi_2(g)$ for all $g \in G$. Now $L(\chi) = \{-1, 0, 2\}$ implies that $\chi_1(g) \in \{0, \pm1, 2\}$ and $\chi_2(g) \in \{0, \pm1\}$ for all $g \in G \setminus Z(G)$.

(2) By Lemma 2.6, we may assume that z is the unique nontrivial element of $Z(G)$. Since by Remark 2.7 we have $z \in \ker(\chi_1)$, it follows that $Z(G) \leqslant \ker(\chi_1)$. Suppose, for a contradiction, that there exists an element $x \in \ker(\chi_1) \setminus Z(G)$. As in the proof of part (1), $\chi(x) + \chi(xz) = 2\chi_1(1)$ and so regarding $L(\chi)$ we have $\chi_1(1) \in \{1, 2\}$. On the other hand, by Remark 2.7 and Lemma 2.2, $\chi(z) = \chi_1(1) - \chi_2(1)$ and $\chi(z) \in \{0, 2\}$. Hence, $\chi_2(1) \in \{1, 2\}$. Since $n \geqslant 4$ is even, the only possibility is $(\chi_1(1), \chi_2(1)) = (2, 2)$. Therefore, $n = 4$ and $|G| = 40$. It is easy to check all groups of order 40 by GAP (see [6]) to see that none of them has the requested property, a contradiction. Hence, $\ker(\chi_1) = Z(G)$.

(3) First we show that $\ker(\chi_2) \leqslant Z(G)$. Suppose, for a contradiction, that there exists an element $x \in \ker(\chi_2) \setminus Z(G)$. Then as in the proof of part (1) for the unique nontrivial element $z \in Z(G)$ we have $\chi(x) - \chi(xz) = 2\chi_2(1)$ and so $\chi_2(1) = 1$. Since $\chi(z) = \chi_1(1) - \chi_2(1) \in \{0, 2\}$ by Remark 2.7 and Lemma 2.2, it follows that $\chi_1(1) \in \{1, 3\}$. Since $n \geqslant 4$ is even, it follows that $(\chi_1(1), \chi_2(1)) = (3, 1)$, $n = 4$ and $|G| = 40$. But $\chi_1(1) = 3$ must divide $|G|$, a contradiction. It follows that $\ker(\chi_2) \leqslant Z(G)$. If $\ker(\chi_2) = Z(G)$, then by part (2) we have $Z(G) = \ker(\chi_1) \cap \ker(\chi_2) = \ker(\chi) = 1$, a contradiction. Hence, Lemma 2.6 implies that $\ker(\chi_2) = 1$. □

Lemma 2.9. *If n is even and z is the nontrivial element of $Z(G)$, then $\chi(z) = 0$.*

Proof. Let $n = \chi_1(1) + \chi_2(1) = 2k$ for a positive integer k. Therefore, by Lemmas 2.2 and 2.6, $\chi(z) \in \{0, 2\}$ for the nontrivial element $z \in Z(G)$. Suppose that $\chi(z) = 2$. On the other hand, $\chi(z) = \chi_1(1) - \chi_2(1)$, by Remark 2.7. Therefore, $\chi_1(1) = k+1$ and $\chi_2(1) = k-1$. Note that $\chi_1(1) \mid |G : Z(G)|$, by [7], Theorem 6.15. Using Lemma 2.6, it follows that $k+1$ is a divisor of $4k^3 - 2k^2 - 2k$ and so $k+1 \mid 4$. Therefore, $k = 1, 3$. Note that $n = 2k \geqslant 3$. Hence, $k = 3$, $n = 6$ and $|G| = 168$. Now it is easy to check all groups of order 168 by GAP (see [6]) to see that the groups of order 168 have no sharp character of type $L = \{-1, 0, 2\}$ with the requested property, a contradiction. Therefore, $\chi(z) = 0$. □

Lemma 2.10. *If n is odd and $Z(G) > 1$, then $G \cong D_{12}$.*

Proof. Let $n = 2k+1$ for a positive integer k. Therefore, by Lemmas 2.2 and 2.6, $|Z(G)| = 2$ and $\chi(z) = -1$ for the nontrivial element $z \in Z(G)$. On the other hand, $\chi(z) = \chi_1(1) - \chi_2(1)$, by Remark 2.7. Therefore, $\chi_1(1) = k$ and $\chi_2(1) = k+1$ are divisors of $|G| = 8k^3 + 8k^2 - 2k - 2$. Hence, $k = 1, 2$.

If $k = 1$, then $|G| = 12$. Now using GAP (see [6]), it is easy to see that $G \cong D_{12}$. If $k = 2$, then $|G| = 90$. By using GAP (see [6]), one can see that the groups of order 90 have no sharp character of type $L = \{-1, 0, 2\}$ with the requested property. □

Proof of the Main Theorem. By Lemma 2.3, χ is the sum of two distinct real valued irreducible characters χ_1 and χ_2 of G. First suppose $n = \chi_1(1) + \chi_2(1) = 2k$ for a positive integer k. By Remark 2.7 and Lemmas 2.6 and 2.9, we have $\chi(z) = \chi_1(1) - \chi_2(1) = 0$ for the unique nontrivial element $z \in Z(G)$. Therefore, $\chi_2(z) = -\chi_2(1) = -k$. On the other hand, $\chi_2(g) \in \{0, \pm 1\}$ for all $g \in G \setminus Z(G)$, by Lemma 2.8. Hence, $L(\chi_2) \subseteq \{0, \pm 1, -k\}$ and by [5], Theorem 1.1, $|G| \mid \prod_{l \in L(\chi_2)} (\chi_2(1) - l)$. Thus, $2k(2k+1)(2k-2) \mid 2k^2(k^2-1)$. Therefore, $4k+2 \mid k(k+1)$. It is easy to see that $(4k+2, k) = 1$ or $(4k+2, k+1) = 1$. Hence, $4k+2 \mid k$ or $4k+2 \mid k+1$, which is a contradiction. Thus, n is odd, and Lemma 2.10 completes the proof. □

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