Bulletin of the Iranian Mathematical Society (2022) 48:3809–3821
https://doi.org/10.1007/s41980-022-00718-8

ORIGINAL PAPER

Groups with Sharp Character of Type {−1, 1, 3}

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Received: 20 January 2022 / Revised: 31 May 2022 / Accepted: 8 July 2022 /
Published online: 6 September 2022

Abstract

For a finite group G and its character χ, let L_χ be the image of χ on $G-\{1\}$. The pair (G, χ) is said to be sharp of type L if $|G| = \Pi_{a\in L}(\chi(1)-a)$, where $L = L_\chi$. The pair (G, χ) is said to be normalized if the principal character of G is not an irreducible constituent of χ. In this paper, we study normalized sharp pairs of type $L = \{-1, 1, 3\}$ proposed by Cameron and Kiyota in [J Algebra 115(1):125–143, 1988], under some additional hypotheses.

Keywords Sharp character · Finite group · Normalized pair

Mathematics Subject Classification 20C15 · 20B05

Communicated by Mohammad Shahryari.

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1 Introduction

Let G be a finite group with center $Z(G)$. It is known that for any complex character χ of G, the order of G divides $\Pi_{a \in L_\chi}(\chi(1) - a)$, where $L_\chi := \{\chi(g) \mid 1 \neq g \in G\}$ (see [4, 12]). The pair (G, χ) (or briefly, the character χ) is called sharp of type L or L-sharp if $|G| = \Pi_{a \in L}(\chi(1) - a)$, where $L = L_\chi$. Clearly, χ is faithful whenever (G, χ) is L-sharp. The pair (G, χ) is said to be normalized if $\langle \chi, 1_G \rangle = 0$.

The notion of sharpness was first introduced for permutation characters by Ito and Kiyota in [11], as a generalization of sharply t-transitive permutation representations. Cameron and Kiyota extended this concept to arbitrary group characters and posed the problem of determining all the L-sharp pairs (G, χ) for a given set L ([5]). The case that L_χ contains an irrational number has been settled by Alvis and Nozawa in [3]. The case $L_\chi = \{l\}$, where l is a rational integer, and the case $L_\chi = \{-1, 1\}$ are determined in [5, 6]. The sharp pairs of type $\{-1, 2\}$ and $\{-2, 1\}$ with non-trivial centers have been studied by Nozawa and Uno in [16]. In [18], Yogochi has classified the finite groups with sharp characters of type $\{l, l + p\}$ for an odd prime p, under the additional hypotheses $Z(G) > 1$ and $\langle \chi, \chi \rangle = p$. Moreover, the sharp pairs of type $\{-1, 3\}$ and $\{-3, 1\}$ has been studied in [2]. In [1], Abdollahi et al. proved that if (G, χ) is a normalized sharp pair and L_χ contains at least one irrational value, then the inner product $\langle \chi, \chi \rangle$ is uniquely determined by L_χ.

In Problem 7.5 of [5], it is proposed to find finite groups G having a sharp character of type $\{-1, 1, 3\}$. If (G, χ) is sharp of type $\{-1, 1, 3\}$, then $\langle \chi, \chi \rangle \leq 3$ and so $\langle \chi, \chi \rangle \in \{1, 2, 3\}$ (see [5, Proposition 1.3(ii)]). Since $\chi(g) = 3$ for some $g \in G$ and $\chi(1) > 1$, it follows from [10, Theorem 3.15] that χ is not irreducible. Thus, $\langle \chi, \chi \rangle \in \{2, 3\}$. Therefore, χ is the sum of two or three distinct irreducible characters of G. In this paper, we consider the case that $\langle \chi, \chi \rangle = 2$, and characterize solvable groups with non-trivial center having sharp character χ of type $\{-1, 1, 3\}$. In fact, we prove the following theorem:

Main Theorem: Let G be a finite solvable group with a normalized sharp character χ of type $\{-1, 1, 3\}$. If $Z(G) \neq 1$ and $\langle \chi, \chi \rangle = 2$, then one of the following cases occur:

1. G is isomorphic to one of the groups $C_2 \times S_4$ or $(((C_2 \times D_8) : C_2) : C_3) : C_2$;
2. $|G| = 8(k-1)k(k+1)$, where $k = 2^l - 1$ for some integer $l \geq 3$ and $Z(G) \cong C_2$ is the unique minimal normal subgroup of G. Moreover, if $3 \nmid k$, then the prime graph of $G/Z(G)$ is disconnected.

2 Preliminary Results

Suppose that (G, χ) is a normalized sharp pair with $L_\chi = \{-1, 1, 3\}$. Let $n := \chi(1)$. Since χ is sharp, $|G| = (n-1)(n+1)(n-3)$ and $n \geq 4$. Note that $\langle \chi, \chi \rangle \in \{2, 3\}$, so we shall distinguish as $\chi = \chi_1 + \chi_2$ or $\chi = \chi_1 + \chi_2 + \chi_3$, for some pairwise distinct irreducible characters χ_1, χ_2 and χ_3 of G.

Lemma 2.1 *Let $\ell = \langle \chi, \chi \rangle$. Then*

(1) $Z(G) = \cap_{i=1}^{\ell} Z(\chi_i)$.

(2) *$Z(G)$ is the direct product of at most ℓ cyclic subgroups.*

Proof Since χ is faithful, the intersection of kernels of irreducible constitutes of χ is trivial. Now, (1) follows from the proof of [10, Corollary 2.28]. Moreover, from (1), we conclude that G can be embedded into $\Pi_{i=1}^{\ell}\frac{G}{\ker(\chi_i)}$ and so $Z(G)$ is isomorphic to a subgroup of $\Pi_{i=1}^{\ell}Z\big(\frac{G}{\ker(\chi_i)}\big)$. Now, from [10, Lemma 2.27 (f)], we have

$$Z(G) \hookrightarrow \Pi_{i=1}^{\ell}\frac{Z(\chi_i)}{\ker(\chi_i)}.$$

Now, [10, Lemma 2.27 (d)] completes the proof. □

Lemma 2.2 *Let p, q and r be arbitrary odd prime divisors of $n-1$, $n+1$, and $n-3$, respectively. Then*

(1) *If $g \in G$ is of order p, then $\chi(g) = 1$.*
(2) *If $h \in G$ is of order q, then $\chi(h) = -1$.*
(3) *If $x \in G$ is of order r, then $\chi(x) = 3$.*
(4) *There is no element of order pq, pr, or qr in G.*

Proof We use the fact that, if s is a prime, θ is a rational valued character of a finite group G, and $y \in G$ then $\theta(y^s) \equiv \theta(y) \pmod s$ [18,Proposition 3(1)].

(1) Since $\chi(g) \equiv \chi(1) = n \pmod p$, and p does not divide $(n+1)(n-3)$, it follows that $\chi(g) = 1$.
(2) Since $\chi(h) \equiv \chi(1) = n \pmod q$, and q does not divide $(n-1)(n-3)$, it is obvious that $\chi(h) = -1$.
(3) Since $\chi(x) \equiv \chi(1) = n \pmod r$, and r does not divide $(n-1)(n+1)$, we get that $\chi(x) = 3$.
(4) Let y be an element of order pq. Then, $\chi(y) \equiv \chi(y^q) \pmod q$ and so by (1) $\chi(y) \equiv 1 \pmod q$. Thus, q divides $\chi(y) - 1$, and since q is odd and $\chi(y) \in \{-1, 1, 3\}$, it follows that $\chi(y) = 1$. Now, (2) implies that $\chi(y) \equiv \chi(y^p) = -1 \pmod p$ and so p divides $\chi(y) + 1$. Thus, $\chi(y) = -1$ as p is odd and $\chi(y) \in \{-1, 1, 3\}$. This is a contradiction. A similar argument shows that G has no element of order pr or qr.

□

Lemma 2.3 (1) *If G is solvable, then at least one of the integers $n-1$, $n+1$, or $n-3$ is a 2-power.*
(2) *n is odd.*

Proof (1) Suppose on the contrary that there exist odd primes p, q and r dividing $n-1$, $n+1$, and $n-3$, respectively. By Lemma 2.2 (4), G has no element of order pq, pr, or qr. This contradicts [14].

(2) If n is even, then $|G|$ is odd and so G is solvable by Feit–Thompson odd order Theorem ([7]). This contradicts part (1). □

Lemma 2.4 *If $n \neq 5, 7$, then at most one of the integers $n-1$, $n+1$, or $n-3$ is a 2-power.*

Proof Note that for $n \geq 9$, the smallest 2-power greater than or equal to $n-3$ is 8, and for integers greater than or equal to 8, the difference in 2-powers is at least 8. Since the difference between $n+1$ and $n-3$ is 4, the set $\{n-3, n-1, n+1\}$ includes at most one 2-power. □

Corollary 2.5 *If G is solvable and $n \neq 5, 7$, then exactly one of the integers $n-1$, $n+1$, or $n-3$ is a 2-power.*

Proof Using Lemma 2.3(1) and Lemma 2.4, we are done. □

Remark 2.6 By Lemma 2.3(2), we may assume that $n = 2k+1$, for some integer $k \geq 2$. Thus, $|G| = 8(k-1)k(k+1)$.

Lemma 2.7 *If $k = 2$, then $G \cong C_2 \times S_4$, where S_4 is the symmetric group of order* 4.

Proof Let $k = 2$. Then, by Remark 2.6, $|G| = 48$. Using GAP [17] shows that $G \cong C_2 \times S_4$. □

Lemma 2.8 *Suppose that there exists $i \in \{-1, 1, 3\}$, such that G has a non-trivial normal p-subgroup P for some odd prime $p \in \pi(n-i)$. Then, for all odd primes $q \in \bigcup_{j \in \{-1,1,3\} \setminus \{i\}} \pi(n-j)$, every Sylow q-subgroup of G is cyclic.*

Proof Take an arbitrary Sylow q-subgroup Q of G, where q is an odd prime in $\bigcup_{j \in \{-1,1,3\} \setminus \{i\}} \pi(n-j)$. Consider the subgroup $H = PQ$. Then, H is solvable and Lemma 2.2(4) shows that every element of H is a prime power order. Now, it follows from [9, Theorem 1] that $H/P \cong Q$ is cyclic, as desired. □

3 Proof of the Main Theorem

Throughout this section, we assume that (G, χ) is a normalized sharp pair of type $\{-1, 1, 3\}$, such that $\langle \chi, \chi \rangle = 2$. Then, $\chi = \chi_1 + \chi_2$ for distinct irreducible characters χ_1 and χ_2 of G.

Lemma 3.1 *χ_1 and χ_2 are real valued.*

Proof Since χ is rational valued, we conclude that $\chi_1 + \chi_2 = \overline{\chi_1} + \overline{\chi_2}$. As complex conjugate of an irreducible character is also irreducible and irreducible characters are linearly independent, it follows that either $\overline{\chi_1} = \chi_2$, or both χ_1 and χ_2 are real valued. If $\overline{\chi_1} = \chi_2$, then $\chi(1) = 2\chi_1(1)$ which contradicts Lemma 2.3. Therefore, the result follows. □

Lemma 3.2 (1) *The center $Z(G)$ of G is an elementary abelian 2-group of order at most* 4.

(2) *If z is a non-trivial element of $Z(G)$, then*

$$(\chi_1(z), \chi_2(z)) \in \{(\chi_1(1), -\chi_2(1)), (-\chi_1(1), \chi_2(1))\}.$$

Proof It follows from Lemma 2.1(1) that $Z(G) = Z(\chi_1) \cap Z(\chi_2)$. Since both χ_1 and χ_2 are real valued, it follows from [10, Lemma 2.27(c)] that $\chi_i(z) = \pm\chi_i(1)$ and so $\chi_i(z^2) = \chi_i(1)$, for all $z \in Z(G)$ and $i \in \{1, 2\}$. Thus, $\chi(z^2) = \chi_1(z^2) + \chi_2(z^2) = \chi(1)$ and thus $z^2 = 1$, since χ is faithful. Now, Lemma 2.1(2) completes the proof of (1). Let z be a non-trivial element of $Z(G)$. Note that $\chi_i(z) = \pm\chi_i(1)$ for $i = 1, 2$, by [10, Lemma 2.27(c)]. Since $\chi(z) = \chi_1(z) + \chi_2(z) \geq -1$ and $\chi(z) \neq \chi(1) = \chi_1(1) + \chi_2(1)$ (as χ is faithful), it follows that $(\chi_1(z), \chi_2(z)) = (\chi_1(1), -\chi_2(1))$, or $(\chi_1(z), \chi_2(z)) = (-\chi_1(1), \chi_2(1))$. This completes the proof of (2). □

Lemma 3.3 $|Z(G)| \leq 2$.

Proof We first prove that there exists at most one element $z \in Z(G)$, such that $(\chi_1(z), \chi_2(z)) = (\chi_1(1), -\chi_2(1))$. Suppose that there exist elements z_1 and z_2 of $Z(G)$, such that

$$(\chi_1(z_1), \chi_2(z_1)) = (\chi_1(z_2), \chi_2(z_2)) = (\chi_1(1), -\chi_2(1)).$$

For $i = 1, 2$, we have $\chi_i(z_1z_2) = \lambda_i(z_1z_2)\chi_i(1)$ for some linear character λ_i of $Z(G)$ ([10, Lemma 2.27(c)]). However, by Lemma 3.2, $Z(G)$ is an elementary abelian 2-group; thus, $\lambda_i(z_1z_2) = \lambda_i(z_1)\lambda_i(z_2) = 1$. It follows that $\chi(z_1z_2) = \chi_1(z_1z_2) + \chi_2(z_1z_2) = \chi_1(1) + \chi_2(1) = \chi(1)$ and so $z_1z_2 = 1$. Thus, $z_1 = z_2$ by Lemma 3.2.

A similar argument shows that there exists at most one element $z' \in Z(G)$, such that $(\chi_1(z'), \chi_2(z')) = (-\chi_1(1), \chi_2(1))$. Now, Lemma 3.2 implies that $|Z(G)| \leq 3$, and since $Z(G)$ is a 2-group, we have $|Z(G)| \leq 2$. This completes the proof. □

Remark 3.4 In the case that $Z(G) \neq 1$, by Lemma 3.2 (2) and Lemma 3.3, without loss in generality, we may assume that there exists a (unique) non-trivial element $z \in Z(G)$, such that $Z(G) = \langle z \rangle$ and $\chi_1(z) = \chi_1(1)$, $\chi_2(z) = -\chi_2(1)$.

Lemma 3.5 *The group G is not nilpotent.*

Proof Suppose on the contrary that G is nilpotent. Since $|G| = 8(k-1)k(k+1)$, there exists an odd prime p dividing $|G|$ and so p divides $|Z(G)|$. This contradicts Lemma 3.2. □

Lemma 3.6 *Let $Z(G) = \langle z \rangle$, for some non-trivial element $z \in G$. Then*

(1) *If $\chi(z) = 1$, then $(\chi_1(1), \chi_2(1)) = (k + 1, k)$;*
(2) *If $\chi(z) = -1$, then $(\chi_1(1), \chi_2(1)) = (k, k + 1)$;*
(3) *If $\chi(z) = 3$, then $(\chi_1(1), \chi_2(1)) = (k + 2, k - 1)$.*

Proof By the equations $2k + 1 = \chi(1) = \chi_1(1) + \chi_2(1)$ and $\chi(z) = \chi_1(1) - \chi_2(1)$, we are done. □

Lemma 3.7 *Let $1 \neq z \in Z(G)$ and $\chi(z) = 3$. Then*

1. $|G| = 7920$ *or* 85008.
2. *G has no normal abelian Sylow* 3*-subgroups.*
3. *If $k = 10$ and N is a minimal normal abelian subgroup of G, then $|N| = 11$ or $N = Z(G)$.*

4. *If* $k = 22$ *and* N *is a minimal normal abelian subgroup of* G*, then* $|N| = 23$ *or* $N = Z(G)$.

Proof (1) By Lemma 3.6(3), we have $\chi_1(1) = k + 2$ and by [10, Theorem 6.15], $\chi_1(1) \mid |G : Z(G)|$. Therefore, $k + 2 \mid 4(k - 1)k(k + 1)$ and

$$\frac{4(k-1)k(k+1)}{k+2} = 4k^2 - 8k + 12 - \frac{24}{k+2}$$

is an integer. Thus, $k = 0, 1, 2, 4, 6, 10$ or 22. But $n \geq 5$, therefore $k \geq 2$. Thus, $|G| = 48, 480, 1680, 7920$ or 85008.

If $|G| = 48$, then χ must have two irreducible constituents of degrees 1 and 4. On the order hand, by Lemma 2.7, only group of order 48 with sharp character of type $\{-1, 1, 3\}$ is $C_2 \times S_4$. However, one can see that this group has not irreducible character of degree 4.

If $|G| = 480$, then by Lemma 2.2, G does not have an element of order 15. Moreover, $|Z(G)| = 2$, and by Lemma 3.1, χ must have two irreducible constituents of degrees 3 and 6 with integer values. Using GAP (see the following commands) shows that there is no group of order 480 with sharp character of type $\{-1, 1, 3\}$ with the mentioned properties.

```
b:=AllSmallGroups(480,IsNilpotent,false);;
F:=Filtered(b,i->Size(Center(i))=2);;
of:=List(F,i->Set(List(i,j->Order(j))));;
c:=Filtered(of,i->Size(Intersection(i,[15]))=0);;
S:=Filtered(F,i->Set(List(i,j->Order(j)))=c[1]);
Irr(S[1]);
```

If $|G| = 1680$, then by Corollary 2.5, G is nonsolvable and Lemma 2.2 shows that G does not have an element of order 15 or 21 or 35. Using GAP shows that each nonsolvable group of order 1680 has at least an element of order 15 or 21 or 35.

(2) Suppose on the contrary that G has a normal abelian Sylow 3-subgroup of order 3^f, where $f \in \mathbb{N}$. By Theorem 6.15 in [10], we have

$$\chi_2(1) = k - 1 \mid \frac{8(k-1)k(k+1)}{3^f};$$

thus, $\frac{8k(k+1)}{3^f}$ is integer. Therefore, $3 \mid k$ or $3 \mid k + 1$. Then, $3 \nmid k - 1$. However, from (1), $k - 1 = 9$ or $k - 1 = 21$, a contradiction.

(3) Suppose N is a minimal normal abelian subgroup of G, such that $|N| \neq 11$. Then, N is a p-group and $\chi_i(1)| \mid G : N \mid$ for $i = 1, 2$. If $p = 3$, then by (2), we have $|N| = 3$. Therefore, $k - 1 = 9 \mid 2^4.3.5.11$, a contradiction. If $p = 5$, then $|N| = 5$. Suppose H is a Sylow 11-subgroup of G. Then, H is a Sylow 11-subgroup of HN. Therefore, $11s + 1 \mid |N| = 5$. Thus, $s = 0$ and HN has only one Sylow 11-subgroup. Then, $HN = H \times N \cong C_5 \times C_{11} \cong C_{55}$ and G has an element of order 55. This contradicts Lemma 2.2(4). If $p = 2$, then $|N| = 2^\alpha$ and $\chi_1(1) = 12 \mid |G : N| = 2^{4-\alpha}.9.5.11$. Therefore, $\alpha = 1$ or 2. If $\alpha = 2$, $|N| = 4$

and since N is a minimal normal abelian subgroup of G, we have $Z(G) \bigcap N = 1$. Therefore, $Z(G)N = Z(G) \times N$ and $|Z(G)N| = 8$. Therefore, $\chi_1(1) = 12 \mid |G : Z(G)N| = 2.9.5.11$, a contradiction. Thus, $\alpha = 1$ and $|N| = 2$. Then, N is a central subgroup of G and, therefore, $N = Z(G)$.

(4) Suppose that N is a minimal normal abelian subgroup of G, such that $|N| \neq 23$. Then, N is a p-group and $\chi_i(1)| \mid G : N \mid$ for $i = 1, 2$. By part (2), $p \neq 3$. If $p = 7$, then $|N| = 7$ and $\chi_2(1) = 21 \mid 16.3.11.23$, a contradiction. If $p = 11$, then $|N| = 11$. Let H be a Sylow 23-subgroup of G. Then, H is a Sylow 23-subgroup of HN. So $23s + 1 \mid |N| = 11$. Thus, $s = 0$ and HN has only one Sylow 23-subgroup. Hence, $HN = H \times N \cong C_{11} \times C_{23} \cong C_{253}$ and G has an element of order 253. This contradicts Lemma 2.2(4). If $p = 2$, then $|N| = 2^\alpha$ and $\chi_1(1) = 24 \mid 2^{4-\alpha}.3.7.11.23$. Therefore, $\alpha = 1$ and $|N| = 2$. Thus, N is a central subgroup of G and so $N = Z(G)$. □

Lemma 3.8 *Suppose that $Z(G)$ is non-trivial. Then*

(1) $\chi_1(g) \in \{0, \pm 1, 2, 3\}$ *and* $\chi_2(g) \in \{0, \pm 1, \pm 2\}$ *for all* $g \in G \setminus Z(G)$.
(2) *If* $n \neq 5, 7$, *then* $\ker(\chi_1) = Z(G)$.
(3) *If* $\ker(\chi_2) \nleq Z(G)$, *then* $G \cong C_2 \times S_4$, *where* S_4 *is the symmetric group of degree* 4.
(4) *If* $G \ncong C_2 \times S_4$, *then* $\ker(\chi_2) = 1$.
(5) *If* $n \neq 5, 7$, *then* $Z(\chi_1) = Z(\chi_2) = Z(G)$.

Proof By Remark 3.4, assume that there exists a non-trivial element z of $Z(G)$, such that $\chi_1(z) = \chi_1(1)$, $\chi_2(z) = -\chi_2(1)$. Therefore, using Lemma 3.3, $Z(G) \leq \ker(\chi_1)$.

(1) Note that if $\mathcal{X}_i$ is a representation corresponding to χ_i, for $i \in \{1, 2\}$, then $\mathcal{X}_1(z) = I_{\chi_1(1)}$ and $\mathcal{X}_2(z) = -I_{\chi_2(1)}$ by [10, Lemma 2.27]. Therefore, $\chi(gz) = \chi_1(g) - \chi_2(g)$ for all $g \in G$. Thus, $\chi(g) + \chi(gz) = 2\chi_1(g)$ and $\chi(g) - \chi(gz) = 2\chi_2(g)$ for all $g \in G$. It follows that $\chi_1(g) \in \{0, \pm 1, 2, 3\}$ and $\chi_2(g) \in \{0, \pm 1, \pm 2\}$ for all $g \in G \setminus Z(G)$.

(2) Suppose on the contrary that there exists $x \in \ker(\chi_1) \setminus Z(G)$. Then, by part (1), $\chi_1(1) = \chi_1(x) \in \{1, 2, 3\}$. Since $\chi(z) = \chi_1(1) - \chi_2(1)$, it follows that $(\chi_1(1), \chi_2(1)) \in \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4)\}$. Thus, $n = \chi(1) \leq 7$, a contradiction. Therefore, $\ker(\chi_1) \leq Z(G)$, and we are done.

(3) Let $x \in \ker(\chi_2) \setminus Z(G)$. Then, by part (1), $\chi(x) - \chi(xz) = 2\chi_2(1)$ and $\chi_2(1) \in \{1, 2\}$. Since $\chi(z) = \chi_1(1) - \chi_2(1)$, it follows that:

$$(\chi_1(1), \chi_2(1)) \in \{(2, 1), (4, 1), (3, 2), (5, 2), (1, 2)\}.$$

As $n > 4$, $(\chi_1(1), \chi_2(1)) \in \{(4, 1), (3, 2), (5, 2)\}$. If $(\chi_1(1), \chi_2(1)) = (5, 2)$, then $|G| = 192$ and $\chi_1(1) = 5$ must divide $|G|$, a contradiction. It follows that $(\chi_1(1), \chi_2(1)) \in \{(4, 1), (3, 2)\}$. Thus, $n = 5$, so that $|G| = 48$. Now, using GAP [17], one can see that $G \cong C_2 \times S_4$.

(4) By (3), $\ker(\chi_2) \leq Z(G)$. It follows from Lemma 3.3 that $\ker(\chi_2) = Z(G)$ or 1. Suppose, for a contradiction, $\ker(\chi_2) = Z(G)$. Since $Z(G) \leq \ker(\chi_1)$, we get $1 = \ker(\chi) = \ker(\chi_1) \cap \ker(\chi_2) = Z(G)$, a contradiction. Hence, $\ker(\chi_2) = 1$.

(5) It follows from [10, Lemma 2.27(f)] that $Z(\chi_2)/\ker(\chi_2) = Z(G/\ker(\chi_2))$. By (4), $Z(\chi_2) = Z(G)$. Now, we put $A = \{g \in G \mid \chi_1(g) = -\chi_1(1)\}$. Let $g \in A$.

From (2), $g \notin Z(G)$. Thus, $\chi_1(g) = -\chi_1(1)$ and $\chi_2(g) \in \{0, \pm 1, \pm 2\}$. We have $\chi_1(1) = \chi_2(g) - \chi(g) \leq 3$. However, $n = 2k+1 \geq 9$, and by Lemma 3.6, $\chi_1(1) \geq k$. This is a contradiction. Therefore, $A = \emptyset$ and $Z(\chi_1) = Z(G)$. □

Theorem 3.9 *Let $Z(G) = \langle z \rangle$ for some non-trivial element $z \in G$. Then, $\chi(z) \in \{-1, 1\}$.*

Proof Let $\chi(z) = 3$. Then, by Lemma 3.7, we have $|G| = 7920$ ($k = 10$) or $|G| = 85008$ ($k = 22$). Suppose N is a non-abelian minimal normal subgroup of G. Then, $N = S^b$, where S is a non-abelian simple group and b is a positive integer. We consider the following cases:

Case 1: Suppose that $k = 10$. By GAP ([17]), $S \cong A_5, A_6$ or PSL$(2, 11)$. Since $|G| = 7920 = 2^4.3^2.5.11$, we must have $b = 1$ and so N is simple. Using Lemma 3.6, we have $\chi_1(1) = 12$ and $\chi_2(1) = 9$. Moreover, by Lemma 3.8, $\chi_1(x) \in \{0, \pm 1, 2, 3\}$ and $\chi_2(x) \in \{0, \pm 1, \pm 2\}$, for every $x \in N \setminus \{1\}$. For $i = 1, 2$, Clifford's theorem shows that $(\chi_i)_N = m_i \sum_{g \in T_i} \theta_i^g$, where $\theta_i \in \mathrm{Irr}(N)$ and T_i is a transversal for the right cosets of $I_G(\theta_i)$ in G. If $S \cong A_5$, then using the character table of A_5, we obtain a contradiction, since $(\chi_2)_N = m_2 \sum_{g \in T_2} \theta_2^g \in \{0, \pm 1, \pm 2\}$. Similarly, if $S \cong A_6$ or $S \cong$ PSL$(2, 11)$ again, we obtain a contradiction. Hence, all minimal normal subgroups of G are abelian. From Lemma 3.7 and [13,Corollary 1], we conclude that G has exactly two minimal normal subgroups N_1 and N_2, such that $N_1 = Z(G)$ and $N_2 \in Syl_{11}(G)$. Moreover, by Lemma 2.8 and this fact that $N_2 \lhd G$, 3-Sylow subgroups of G are cyclic. Now, let $R(G)$ be the solvable radical of G, $R(G) < M \lhd G$ and $M/R(G)$ be a chief factor of G. Note that by Lemma 2.3, G is nonsolvable. An argument similar to that given above shows that $S := M/R(G) \cong A_5, A_6$ or PSL$(2, 11)$. If $S \cong$ PSL$(2, 11)$, then as $11 \mid |R(G)|$, we have a contradiction. If $S \cong A_6$, then as 3-Sylow subgroups of A_6 are non-cyclic, we obtain a contradiction. Hence, $S \cong A_5$. Let $C/R(G) := C_{G/R(G)}(S)$ and $T := L/R(G)$ be a chief factor of $G/R(G)$ contained in $C/R(G)$. Then, as $S \times T$ is a normal subgroup of G and $S = T \cong A_5$, we have $25 \mid |G|$, this is a contradiction. Thus, $C = R(G)$ and $S \leq G/R(G) \leq \mathrm{Aut}(S)$. Therefore, $33 \mid |R(G)|$, and then, G has an element of order 33 which is impossible.

Case 2: Suppose that $k = 22$. By GAP ([17]), $S \cong$ PSL$(2, 7)$ or PSL$(2, 23)$. Since $|G| = 85008 = 2^4.3.7.11.23$, one can see that $b = 1$ and N is simple. Using Lemma 3.6, $\chi_1(1) = 24$ and $\chi_2(1) = 21$. An argument similar to that given in case (1) shows that all minimal normal subgroups of G are abelian and G has exactly two minimal normal subgroups N_1 and N_2, such that $N_1 = Z(G)$ and $N_2 \in Syl_{23}(G)$. Moreover, if $R(G)$ be the solvable radical of G, $R(G) < M \lhd G$ and $M/R(G)$ be a chief factor of G then, similarly, we have $S := M/R(G) \cong$ PSL$(2, 7)$ or PSL$(2, 23)$. If $S \cong$ PSL$(2, 23)$, then as $23 \mid |R(G)|$, we have a contradiction. Hence $S \cong$ PSL$(2, 7)$. Now, let $C/R(G) := C_{G/R(G)}(S)$ and $T := L/R(G)$ be a chief factor of $G/R(G)$ contained in $C/R(G)$. Since $S \times T$ is a normal subgroup of G and $S = T \cong$ PSL$(2, 7)$, we get $49 \mid |G|$. This is a contradiction. Then, $C = R(G)$ and $S \leq G/R(G) \leq \mathrm{Aut}(S)$. Hence, $G/R(G) \cong S$ and $|R(G)| = 2.11.23$, and $|R(G)/Z(G)| = 11.23$. So $R(G)/Z(G) \cong C_{11} \times C_{23}$ or $C_{11} : C_{23}$. If $R(G)/Z(G) \cong C_{11} \times C_{23}$, then G has an element of order 11×23 which is impossible. Moreover, if $R(G)/Z(G) \cong C_{11} : C_{23}$, then by using the character table of $C_{11} : C_{23}$, we have a contradiction. □

Remark 3.10 Let $\bar{G} = G/Z(G)$ and $n \neq 5, 7$. By Lemma 3.8, $Z(\chi_1) = ker(\chi_1) = Z(G)$, and so, [10, Lemma 2.27 (f)] implies that $Z(\bar{G}) = Z(\chi_1)/ker(\chi_1) = 1$.

Lemma 3.11 *Let $1 \neq z \in Z(G)$ and $k \neq 2, 3$.*

(1) *If $\chi(z) = 1$, then k is even.*
(2) *If $\chi(z) = -1$, then $k - 1$ and k are not 2-powers.*
(3) $4 \nmid k - 1$.

Proof (1) By Lemma 3.6, $\chi_2(1) = k$. We put $L := L_{\chi_2} = \{\chi_2(g) \mid 1 \neq g \in G\}$. By Lemma 3.8, $\chi_2(g) \in \{0, \pm 1, \pm 2, \}$ for $g \in G \setminus Z(G)$. Hence, $L \subseteq \{0, \pm 1, \pm 2, -k\}$. Since $|G| \mid \prod_{l \in L}(\chi_2(1) - l)$, we have $8(k-1)k(k+1)$ is a divisor of $2k^2(k-1)(k+1)(k-2)(k+2)$, and so, $4 \mid k(k-2)(k+2)$. Thus, k is even.

(2) Set $\bar{G} = G/Z(G)$. By Lemma 3.3, $|\bar{G}| = 4(k-1)k(k+1)$. If $\chi(z) = -1$, then by Lemma 3.6, $\chi_1(1) = k$. We put $L := L_{(\chi_1)_{\bar{G}}} = \{(\chi_1)_{\bar{G}}(g) \mid 1 \neq g \in \bar{G}\}$. By Lemma 3.8, $\chi_1(g) \in \{0, \pm 1, 2, 3\}$ for $g \in G \setminus Z(G)$. Hence, $L \subseteq \{0, \pm 1, 2, 3\}$. Since $|\bar{G}| \mid \prod_{l \in L}(\chi_1(1) - l)$, we have $4(k-1)k(k+1)$ divides $k(k-1)(k+1)(k-2)(k-3)$ and so $4 \mid (k-2)(k-3)$. Suppose that $k - 1$ is a 2-power. Then, $k - 1 = 2^q$ for some $1 \neq q \in \mathbb{N}$. Hence, $k - 2$ is odd, and $4 \mid k - 3 = 2^q - 2 = 2(2^{q-1} - 1)$, a contradiction. Now, suppose that $k = 2^q$, for some $1 \neq q \in \mathbb{N}$. Then, $k - 3$ is odd, and $4 \mid k - 2 = 2(2^{q-1} - 1)$, a contradiction.

(3) Suppose $k - 1$ is even. Thus, by the proof of Theorem 3.9 and part (1), $\chi(z) = -1$, and so, $\chi_1(1) = k$. Then, an argument similar to that given in (2) shows that $4 \mid k - 3$. However, $(k-3, k-1) = 2$; thus, $4 \nmid k - 1$. □

Lemma 3.12 *Let $1 \neq z \in Z(G)$, $\bar{G} = G/Z(G)$ and $k \neq 2$ be even. If $p \neq 3$ is an odd prime divisor of $k + 1$ or $k - 1$, then there is no element of order $2p$ in $\bar{G}$.*

Proof By Remark 3.10, $Z(\bar{G}) = 1$. Hence, by Lemma 3.8, $\chi_1(g) \in \{0, \pm 1, 2, 3\}$ for $g \in \bar{G} - \{1\}$. Suppose on the contrary that there exists $g \in \bar{G}$, such that $o(g) = 2p$, where $3 \neq p$ is an odd prime divisor of $k + 1$ or $k - 1$.

First, suppose that $p \mid k+1$. If $\chi(z) = 1$, then $k+1 = \chi_1(1) = \chi_1(g^{2p}) \equiv \chi_1(g^2) \pmod p$. Therefore, $\chi_1(g^2) = 0$. Moreover, $\chi_1(g^2) \equiv \chi_1(g) \pmod 2$. Hence, $\chi_1(g) \in \{0, 2\}$. On the other hand, $k + 1 = \chi_1(g^{2p}) \equiv \chi_1(g^p) \pmod 2$. Since $k + 1$ is odd, $\chi_1(g^p) \in \{\pm 1, 3\}$. However, $\chi_1(g^p) \equiv \chi_1(g) \pmod p$, a contradiction. If $\chi(z) = -1$, we have $k = \chi_1(1) = \chi_1(g^{2p}) \equiv \chi_1(g^2) \pmod p$. Then, $\chi_1(g^2) = -1$. Also, $\chi_1(g^2) \equiv \chi_1(g) \pmod 2$. Thus, $\chi_1(g) \in \{\pm 1, 3\}$. Also, $k = \chi_1(g^{2p}) \equiv \chi_1(g^p) \pmod 2$. Since k is even, it follows that $\chi_1(g^p) \in \{0, 2\}$. Also, $\chi_1(g^p) \equiv \chi_1(g) \pmod p$, which is a contradiction.

Now, suppose that $p \mid k-1$. If $\chi(z) = 1$, then $k+1 = \chi_1(1) = \chi_1(g^{2p}) \equiv \chi_1(g^2) \pmod p$. Therefore, $\chi_1(g^2) = 2$. Moreover, $\chi_1(g^2) \equiv \chi_1(g) \pmod 2$. Thus, $\chi_1(g) \in \{0, 2\}$. Note that $k + 1 = \chi_1(g^{2p}) \equiv \chi_1(g^p) \pmod 2$. Since $k + 1$ is odd, $\chi_1(g^p) \in \{\pm 1, 3\}$. However, $\chi_1(g^p) \equiv \chi_1(g) \pmod p$, which is a contradiction. If $\chi(z) = -1$, we have $k = \chi_1(1) = \chi_1(g^{2p}) \equiv \chi_1(g^2) \pmod p$. Therefore, $\chi_1(g^2) = 1$. Since $\chi_1(g^2) \equiv \chi_1(g) \pmod 2$, we get $\chi_1(g) \in \{\pm 1, 3\}$. Now, $k = \chi_1(g^{2p}) \equiv \chi_1(g^p) \pmod 2$. Note that k is even; therefore, $\chi_1(g^p) \in \{0, 2\}$. Also, $\chi_1(g^p) \equiv \chi_1(g) \pmod p$, a contradiction. □

Let G be a finite group. The prime graph of G, denoted by $\Gamma(G)$, is a graph whose vertices are the prime divisors of $|G|$ and two vertices p, q are adjacent if and only if G contains an element of order pq. We denote the number of the connected components of $\Gamma(G)$ by $com(G)$.

Theorem 3.13 ([8]) *If G is solvable with more than two prime graph components, then G is either Frobenius or 2-Frobenius and G has exactly two components, one of which consists of the primes dividing the lower Frobenius complement.*

Lemma 3.14 *Let $Z(G) \neq 1$ and $\bar{G} = G/Z(G)$. If $k \neq 2$ is even, then the prime graph of $\bar{G}$ is disconnected.*

Proof Since $|\bar{G}| = 4(k-1)k(k+1)$, then the result follows by Lemmas 2.2 and 3.12. □

Proposition 3.15 *Let $Z(G) \neq 1$ and $\bar{G} = G/Z(G)$. If $k \neq 2$ is even, then $\bar{G}$ is not a Frobenius group.*

Proof Let $\bar{G}$ be a Frobenius group with kernel H and complement C. By [15, Lemma 5], $\Gamma(H)$ and $\Gamma(C)$ are the connected components of $\Gamma(\bar{G})$, and $\Gamma(H)$ is complete. Also, by Lemma 2.2, if $k+1 \mid |H|$, then $k-1 \nmid |H|$ and conversely. Therefore, we consider the following cases.

Case 1: If $3 \nmid k+1$, then by Lemma 2.2 and the fact that $\Gamma(H)$ is complete; if $k+1 \mid |H|$, then $k-1 \nmid |H|$ and conversely. Also, $\Gamma(C)$ is connected, and by Lemma 3.12, prime divisors of k are not adjacent to the prime divisors of $k+1$. Hence, one of the following holds:

(a) $|H| = 4k(k-1)$ and $|C| = k+1$;
(b) $|H| = k+1$ and $|C| = 4k(k-1)$.

Suppose (a) holds. Then, $k = 2^m$ and $k-1 = 3^l$ for $m, l \in \mathbb{N}$. However, $|C| \mid |H|-1$. Hence, $k = 6$, a contradiction. Now, suppose (b) is true. Hence, $4k(k-1) \mid k$, a contradiction.

Case 2: If $3 \nmid k-1$, then by Lemma 3.12, one of the following holds:

(a) $|H| = 4k(k+1)$ and $|C| = k-1$;
(b) $|H| = k-1$ and $|C| = 4k(k+1)$.

Suppose (a) holds. Then, $k = 2^m$ and $k+1 = 3^l$ for $m, l \in \mathbb{N}$. However, $|C| \mid |H|-1$; hence, $k = 8$ and $|H| = 2^5.3^2$. Also, H is nilpotent. Let P be the Sylow 3-subgroup of H. Since P is a characteristic subgroup of H and H is normal in $\bar{G}$, then P is normal in $\bar{G}$. Also, P is abelian. However, by Lemma 3.11, $\chi(z) = 1$ for $1 \neq z \in Z(G)$ and $\chi_1(1) = k+1$. Hence, $9 = \chi_1(1) \mid |\bar{G} : P|$, a contradiction. Now, suppose (b) is true. Then, $4k(k+1) \mid k-2$, a contradiction. □

Proposition 3.16 *Let $Z(G) \neq 1$ and $\bar{G} = G/Z(G)$. If $k \neq 2$ is even, then $\bar{G}$ is not a 2-Frobenius group.*

Proof Suppose that $\bar{G}$ is a 2-Frobenius group. Then, $\bar{G} = ABC$, where A and AB are normal subgroups of $\bar{G}$, and AB and BC are Frobenius groups with kernels A and

B, respectively. By [15, Lemma 7], $\Gamma(B)$ and $\Gamma(AC)$ are connected components of $\Gamma(\bar{G})$ and are both complete graphs and B is cyclic of odd order. If $3 \mid k$, then by Lemmas 2.2 and 3.12, we get that $com(\bar{G}) \geq 3$, a contradiction. Thus, $3 \mid k+1$ or $3 \mid k-1$.

Let $3 \mid k+1$. Then, $|AC| = 4k(k+1)$ and $|B| = k-1$, because B is cyclic of odd order. Since $\Gamma(AC)$ is complete, $k = 2^m$ and $k+1 = 3^s$ for $m, s \in \mathbb{N}$. Therefore, $k = 8$ and $|B| = 7$. Also, since AB is a Frobenius group with kernel A and complement B, we have $|B| \mid |A|-1$. Moreover, BC is a Frobenius group with kernel B and complement C, so $|C| \mid |B|-1 = 6$. Also, A is nilpotent, and if 3 divides $|A|$, then A has a normal Sylow 3-subgroup Q of order 3 or 9, and QB is a Frobenius group, which is a contradiction, since 7 does not divide $3-1$ or $9-1$. Thus, 9 divides $|C|$; this is a contradiction, since $|C|$ must divide 6.

Let $3 \mid k-1$. Then, $|AC| = 4k(k-1)$ and $|B| = k+1$, because B is cyclic of odd order. Since $\Gamma(AC)$ is complete, $k = 2^m$ and $k-1 = 3^s$ for $m, s \in \mathbb{N}$. Hence, we have $m = 2$ and $k = 4$. Therefore, $|B| = 5$. Also, A is nilpotent, and if 3 divides $|A|$, then A has normal Sylow 3-subgroup Q of order 3 and QB is a Frobenius group which is a contradiction since 5 does not divide $3-1$. Thus, 3 divides $|C|$. However, since BC is a Frobenius group, this would imply that 3 divides $5-1$ which is also a contradiction. □

Lemma 3.17 *Let G be a solvable group and $Z(G) \neq 1$. If $k \neq 2$, then $k = 2^l - 1$ for some integer $l \geq 2$. Moreover, if $l = 2$, then $G \cong (((C_2 \times D_8) : C_2) : C_3) : C_2$.*

Proof It follows from Corollary 3.13, Lemma 3.14, Proposition 3.15, and Proposition 3.16 that k is odd. If $k = 3$, then $k = 2^2 - 1$, and by Remark 2.6, $|G| = 192$. Using GAP [17] shows that $G \cong (((C_2 \times D_8) : C_2) : C_3) : C_2$. Therefore, we may assume $k \geq 5$. By Lemma 3.11(1), we have $\chi(z) \neq 1$ and Theorem 3.9 shows that $\chi(z) = -1$. Now, it follows from Lemma 3.11(2) that $k-1$ and k are not 2-powers. On the other hand, from Corollary 2.5 one of the integers $n-3 = 2(k-1)$, $n-1 = 2k$, or $n+1 = 2(k+1)$ must be a 2-power. Therefore, we conclude that $k+1 = 2^l$ for some $l \geq 3$, and thus, $k = 2^l - 1$. □

Lemma 3.18 *Let $Z(G) \neq 1$, $\bar{G} = G/Z(G)$, and k be odd. If $p \neq 3$ is an odd prime divisor of k, then there is no element of order $2p$ in $\bar{G}$.*

Proof Since $Z(\bar{G}) = 1$, it follows from Lemma 3.8 that $\chi_1(g) \in \{0, \pm 1, 2, 3\}$ for every $g \in \bar{G} - \{1\}$. Also, by Lemma 3.11, $\chi(z) = -1$ for $1 \neq z \in Z(G)$, and thus, $\chi_1(1) = k$. Suppose there exists $g \in \bar{G}$, such that $o(g) = 2p$, where $p \neq 3$ is an odd prime divisor of k. Then, we have $k = \chi_1(1) = \chi_1(g^{2p}) \equiv \chi_1(g^2) \pmod{p}$. Hence, $\chi_1(g^2) = 0$. Also, $\chi_1(g^2) \equiv \chi_1(g) \pmod{2}$. Hence, $\chi_1(g) \in \{0, 2\}$. Now, $k = \chi_1(g^{2p}) \equiv \chi_1(g^p) \pmod{2}$. However, k is odd; therefore, $\chi_1(g^p) \in \{\pm 1, 3\}$. Since $\chi_1(g^p) \equiv \chi_1(g) \pmod{p}$, we have a contradiction. □

Lemma 3.19 *Let $Z(G) \neq 1$ and $\bar{G} = G/Z(G)$. If k is odd and $3 \nmid k$, then the prime graph of $\bar{G}$ is disconnected.*

Proof The result follows from Lemma 2.2 and Lemma 3.18. □

Theorem 3.20 *Let G be a solvable group and $Z(G) \neq 1$. If $k \neq 2, 3$, then $Z(G)$ is the unique minimal normal subgroup of G.*

Proof Let N be a minimal normal subgroup of G, such that $N \neq Z(G)$. By Lemma 3.6 and Lemma 3.11, $(\chi_1(1), \chi_2(1)) = (k, k+1)$. Since N is an elementary abelian p-group, it follows from [10,Theorem 6.15] that $\chi_i(1) \mid |G : N|$ for $i = 1, 2$. Therefore, $|N| \mid 8(k-1)$.

If $p \neq 2$, then $p|k-1$ and $p|n-3$. Then, by Lemma 2.2, $\chi(x) = 3$ for $x \in N \setminus \{1\}$. We have $\chi(x) + \chi(xz) = 2\chi_1(x)$ for each $x \in G \setminus Z(G)$; thus, $L_{(\chi_1)_{N\setminus\{1\}}} \subseteq \{1, 2, 3\}$. By Clifford's theorem, $(\chi_1)_N(x) = e\sum_{1\leq i\leq t}\theta_i(x)$ where e is an integer. We put $\sum_{1\leq i\leq t}\theta_i(x) = \lambda$. If $(\chi_1)_N(x) = 1$, then $\lambda = 1/e$. However, λ is an algebraic integer. Therefore, $e = 1$. Since N is abelian, we have $k = \chi_1(1) = \sum_{1\leq i\leq t}\theta_i(1) = t$. Thus, N has at least k distinct irreducible characters. Hence, $|N| \geq k$. However, $|N| \mid k-1$, a contradiction. Hence, $L_{(\chi_1)_{N\setminus\{1\}}} \subseteq \{2, 3\}$. If $(\chi_1)_N(x) = 2$, $\lambda = 2/e$, then $e = 1$ or 2. A similar argument as above shows that $e \neq 1$. Therefore, $e = 2$ and $k = \chi_1(1) = 2\sum_{1\leq i\leq t}\theta_i(1) = 2t$, a contradiction. Thus, $(\chi_1)_N(x) = 3$ for $x \in N \setminus \{1\}$. However, $|N| \mid \Pi_{l\in L_{\chi_1}}(\chi_1(1) - l)$, where $L_{\chi_1} = \{\chi_1(x) \mid 1 \neq x \in N\}$. Therefore, $|N| \mid (\chi_1(1) - 3) = k - 3$, a contradiction, because $|N| \mid k-1$ and $|N|$ is odd. Thus, $p = 2$. Also, $k-1$ is even and by Lemma 3.11, $4 \nmid k-1$; hence, $|N| \mid 16$. Suppose that $|N| = 16$. Since N is minimal normal subgroup of G, $Z(G) \cap N = \{1\}$. Therefore, $Z(G)N = Z(G) \times N$ and so $|Z(G)N| = 32$. However, $k+1 = \chi_2(1) \mid |G : Z(G)N|$, a contradiction. Thus, $|N| \mid 8$.

Since $k = \chi_1(1) = e\sum_{1\leq i\leq t}\theta_i(1) = et$, it follows that e and t are odd. Thus, $\chi_1(x) \neq 0$ for each $x \in N$.

If $e = 1$, then $k = \chi_1(1) = t$. Hence, N has at least k distinct irreducible characters. However, $|N| \mid 8$ and and by Lemma 3.17, $k+1$ is 2-power; therefore, $k = 7$ and $|G| = 2688$. Then, $|\bar{G}| = 1344$. Note that $\bar{G}$ is solvable, and by Lemma 3.5, $\bar{G}$ is not nilpotent. Also, χ_1 is faithful in $\bar{G}$ and $\chi_1(1) = k = 7$. Using GAP [17] shows that there is no group with the mentioned properties.

If $e \neq 1$, then $\chi_1(x) \neq \pm 1$ for each $x \in N$. If $\chi_1(x) = 2$ for some $x \in N$, then $e = 2$, a contradiction. Thus, $\chi_1(x) = 3$ for each $x \in N \setminus \{1\}$ and $e = 3$. Thus, $t = k/3$, $3 \mid k$ and $|N| \mid \chi_1(1) - 3 = k - 3$. Hence, N has at least $k/3$ distinct irreducible characters. Thus, $|N| \geq k/3$. Also, $|N| \mid 8$ and $k+1$ is 2-power; therefore, $k = 3$, a contradiction. □

Acknowledgements The authors would like to thank the anonymous referees for their helpful comments. The work of the third author was supported in part by a grant from school of mathematics, Institute for Research in Fundamental Sciences (IPM).

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