Best Known bounds of the permutation codes under Kendall τ-metric

Definition: A *Permutation Code* of length n is a non-empty subset of S_n, the set of all permutations of $[n] := \{1,2,\dots,n\}$. Given a permutation $\pi := [\pi(1), \pi(2), \dots, \pi(i), \pi(i+1), \dots, \pi(n)] \in S_n$, an *adjacent transposition*, $(i, i+1)$, for some $1 \le i \le n-1$, applied to π will result in the permutation $[\pi(1), \pi(2), \dots, \pi(i+1), \pi(i), \dots, \pi(n)]$. For two permutations $\rho, \pi \in S_n$, the *Kendall τ-distance* between ρ and π, $d_K(\rho, \pi)$, is defined as the minimum number of adjacent transpositions needed to write $\rho\pi^{-1}$ as their product. Note that $d_K(\rho, \pi) \le \binom{n}{2}$ for all $\rho, \pi \in S_n$. The maximum size of a permutation code of length n and minimum Kendall τ-distance at least d is denoted by $P(n, d)$.

In the following, we present the best known bounds of $P(n, d)$. Note that the green color shows the exact value of $P(n, d)$ and the blue and yellow colors show the best known upper and lower bounds of $P(n, d)$, respectively. Also the withe color shows the previously known bounds which are not currently the best one.

Note: There exist some results on the exact value of $P(n, d)$ as follows:

- ✓ $P(n, 1) = n!$.
- ✓ $P(n, 2) = \frac{n!}{2}$.
- ✓ If $2/3\binom{n}{2} < d \le \binom{n}{2}$, then $P(n, d) = 2$ (see [4]).
- ✓ If $n \ge 6$ and $3/5\binom{n}{2} < d \le 2/3\binom{n}{2}$, then $P(n, d) = 4$ (see [2]).

Then, for each $n \ge 3$, we present the best known bounds on $P(n, d)$ for all $3 \le d \le 3/5\binom{n}{2}$.

- $P(3,3) = 2$ (see [5])
- $P(4,3) = 5$ and $P(4,4) = 3$ (see [5])

- $n = 5$ (see [6])

d	3	4	5	6
$P(5,d)$	20	12	6	5

- $n = 6$

d	3	4	5	6	7	8	9
$P(6,d)$	≥ 102[6] ≤ 116[1]	64[6]	26[6]	20[6]	11[6]	7[6]	4[6]

- $n = 7$

d	3	4	5	6	7	8
Lower Bound	588[4]	336[3] 315[2] 294[4]	126[2]-[3] 110[4]	84[2]-[3] 55[4]	42[2]-[3] 34[4]	28[2] 17[4]
Upper Bound	719[4] 716[1]	420[4]	186[4]	120[4]	66[4]	45[4]
d	9	10	11	12		
Lower Bound	15[2] 14[4]	13[3] 12[2] 7[4]	8[2]-[3] 2[2]	7[2]-[3] 2[2]		
Upper Bound	28[4]	21[4]	12[4]	8[4]		

- $n = 8$

d	3	4	5	6	7	8	9	10
Lower Bound	3752[3]	2240[3]	672[2]- [3]	448[3]	168[2]- [3]	115[3]	57[3]	48[2]
	3696[2]	2184[2]		392[2]		112[2]	48[2]	43[3]
Upper Bound	5039[2]	2880[2]	1152[2]	720[2]	363[2]	242[2]	141[2]	99[2]
d	11	12	13	14	15	16		
Lower Bound	26[3]	24[2]	15[3]	14[2]	8[3]-[2]	8[2]		
	24[2]	21[3]	14[2]	12[3]				
Upper Bound	64[2]	47[2]	32[2]	25[2]	10[2]	8[2]		

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