New table of bounds on permutation codes under Kendall τ-metric

Alireza Abdollahi

Department of Pure Mathematics
Faculty of Mathematics and Statistics
University of Isfahan
Isfahan 81746-73441
Iran

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This is a joint work done in CSG Research Group (Code-Scheme-Group)

Figure : right to left: Jafari, Parvaresh, Khatami, Sobhani, Bagherian, Abdollahi

CSG Research group Home Page: https://csg.ui.ac.ir/

Rank Modulation I

In order to overcome the challenges posed by flash memories, the rank modulation scheme was proposed in [A. Jiang, R. Mateescu, M. Schwartz, and J. Bruck, Correcting charge-constrained errors in the rank-modulation scheme, IEEE Trans. Inform. Theory, **56** (2010), 2112-2120. (first appeared in ISIT 2008)]

Figure : right to left: Bruck, Schwartz, Mateescu, Jiang

Rank Modulation II

Definition (Rank Modulation)

Use the relative order of cell levels to represent data.

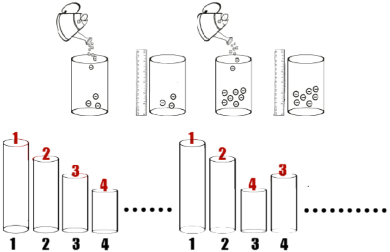
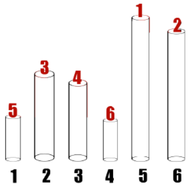


Figure : Figures are taken from (FMS2014_Tutorial_Part3_Jiang.pdf) in A. Jiang's Home Page

Rank Modulation III

Example: Every rank has one cell



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This corresponds to the permutation $[5, 3, 4, 6, 1, 2]$ (representated by array) or $(1, 5)(2, 3, 4, 6)$ as product of cycles.

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Kendall τ-metric on Permutation Codes

Definition

The Kendall distance between two permutations σ and τ denoted by $d_K(\sigma, \tau)$ is the minimum number of adjacent transpositions $(i, i+1)$ such that their product is equal to $\sigma \cdot \tau^{-1}$, where the τ^{-1} is the inverse of τ and the composition $\cdot$ of two permutations is done from the right i.e. the value of $\sigma \cdot \tau^{-1}$ at $\ell \in [n]$ is equal to the value of τ^{-1} at $\sigma(\ell)$.

Example

$d_K([2,1,3,4,5],[1,2,3,5,4]) = d_K((1,2),(4,5)) = 2,$
$d_K([2,3,1,5,4],[2,1,3,5,4]) = d_K((1,2,3)(4,5),(4,5)(1,2)) = 1.$

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Main problem

Main problem of coding theory for PC

Find $P(n,d) := \max\{|C| \mid \varnothing \neq C \subseteq S_n \; d_K(C) \geq d\}$ or find "good" lower or upper bounds for $P(n,d)$. Here $d_K(C) := \min\{d_K(\sigma,\tau) \mid \sigma \neq \tau, \sigma,\tau \in C\}$.

Known values of $P(n, d)$—I

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- $P(3,3) = 2$, $P(4,3) = 5$, $P(4,4) = 3$.
- $P(5,3) = 20$, $P(5,4) = 12$, $P(5,5) = 6$, $P(5,6) = 5$. $P(6,4) = 64$, $P(6,5) = 26$, $P(6,6) = 20$, $P(6,7) = 11$, $P(6,8) = 7$, $P(6,9) = P(6,10) = 4$. [S. Vijayakumaran, Largest permutation codes with the Kendall τ-metric in S_5 and S_6, IEEE Comm. Letters, **20** (2016), No. 10, 1912-1915.]

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The "least" unknown value of $P(n, d)$—II

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- $P(6, 3) \geq 102$. [S. Vijayakumaran, Largest permutation codes with the Kendall τ-metric in S5 and S6, IEEE Comm. Letters, **20** (2016), No. 10, 1912-1915.]
- $P(6, 3) \leq 116$. [A. Abdollahi, J. Bagherian, F. Jafari, M. Khatami, F. Parvaresh and R. Sobhani, New upper bounds on the size of permutation codes with minimum Kendall τ-metric of three, to appear in Cryptogr. Commun.]

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Conjecture

$P(6,3) = 102$. A possible way to attack the conjecture is to solve a specific binary linear programming problem with 720 indeterminates and 720 constraints given in [S. Vijayakumaran, Largest permutation codes with the Kendall τ-metric in S_5 and S_6, IEEE Comm. Letters, **20** (2016), No. 10, 1912-1915.]

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Our main result

Theorem

$P(n,d)=4$ for all $n \geq 6$ and $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$.

Sketch of Proof (Upper bound)

It follows from Theorem 23 of [X. Wang, Y. Zhang, Y. Yang and G. Ge, New bounds of permutation codes under Hamming metric and Kendall's τ-metric, Des. Codes Cryptogr., 85 (2017), No. 3, 533-545.] that if $P(n,d) \geq 5$, then we must have $\binom{5}{2} d \leq 6 \times \binom{n}{2}$ and so $d \leq \frac{3}{5}\binom{n}{2}$. Therefore $P(n,d) \leq 4$.

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We need the following lemma: Since $P(n,d+1) \leq P(n,d)$, it is enough to show that there exists an $P(n, \lfloor \frac{2}{3}\binom{n}{2} \rfloor) \geq 4$ or equivalently show that there exists a subset C of S_n of size 4 such that $d_K(C) \geq \lfloor \frac{2}{3}\binom{n}{2} \rfloor$.

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Sketch of Proof(Lower Bound)

Sketch of Proof (Constructing Permutations)

We need the following lemma: Let $n \geq 5$ be an integer. If $n \equiv 0, 2 \pmod 3$ ($n \equiv 1 \pmod 3$), then there exist 3 non-empty subsets with the same sumset which partitions $[n]$ ($[n] \setminus \{1\}$), respectively.

Sketch of Proof(Lower Bound)

Sketch of Proof of the lemma

- If n is 5, 6, 7, 8, 9 and 10, respectively, then $\big\{\{5\},\{1,4\},\{3,2\}\big\}$,$\big\{\{6,1\},\{5,2\},\{3,4\}\big\}$,$\big\{\{2,7\},\{3,6\},\{4,5\}\big\}$,$\big\{\{8,4\},\{7,3,2\},\{1,5,6\}\big\}$,$\big\{\{6,5,4\},\{9,1,2,3\},\{8,7\}\big\}$ and $\big\{\{10,8\},\{9,2,7\},\{3,4,6,5\}\big\}$ are the partitions of $[n]$ or $[n]\setminus\{1\}$ satisfying the lemma.

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- Now suppose that $n > 10$. Hence there exist $t > 0$ and $r \in \{5,6,7,8,9,10\}$ such that $n = 6t + r$. Note that if $n \equiv 1 \pmod 3$, then $r \in \{7,10\}$.

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- Consider $t+1$ subsets Θ_1,...,Θ_{t+1} of $[n]$ as follows:

$$\underbrace{1,\ldots,r}_{\Theta_1},\underbrace{r+1,\ldots,r+6}_{\Theta_2},\ldots,\underbrace{n-11,\ldots,n-6}_{\Theta_t},\underbrace{n-5,\ldots,n}_{\Theta_{t+1}}$$

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Thanks for your attention