Improved Permutation Arrays for Kendall Tau Metric*

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Abstract

Permutation arrays under the Kendall-τ metric have been considered for error-correcting codes. Given n and $d \in [1, \binom{n}{2}]$, the task is to find a large permutation array of permutations on n symbols with pairwise Kendall-τ distance at least d. Let $P(n, d)$ denote the maximum size of any permutation array of permutations on n symbols with pairwise Kendall-τ distance d. Using new recursive techniques, new automorphisms, and programs that combine randomness and greedy strategies, we obtain several improved lower bounds for $P(n, d)$.

1 Introduction

In [1, 2, 5, 8, 10, 11], permutation arrays under the Kendall-τ metric were studied. This complemented many studies of permutation arrays under other metrics, such as the Hamming metric [3] [4] [6], Chebyshev metric [9] and several others [7]. The use of the Kendall-τ metric was motivated by applications of error correcting codes and rank modulation in flash memories [8].

Let σ and π be two permutations (or strings) over an alphabet $\Sigma \subseteq [1...n] = \{1, 2, ..., n\}$. The *Kendall-$\tau$ distance* between σ and π, denoted by $d(\sigma, \pi)$, is the minimum number of adjacent transpositions (bubble sort operations) required to transform σ into π. For an array (set) A of permutations (strings), the pairwise Kendall-τ distance of A, denoted by $d(A)$, is $\min\{\ d(\sigma, \pi) \mid \sigma, \pi \in A\ \}$. An array A of permutations on $[1...n]$ with $d(A) = d$ will be called a (n, d)-PA. Let $P(n, d)$ denote the maximum cardinality of any (n, d)-PA A.

Vijayakumaran [10] showed several lower bounds for $P(5, d)$ and $P(6, d)$ using integer linear programming. Buzaglo and Etzion [5] showed many new bounds, including that $P(7, 3) \geq 588$ using two permutation representatives and a set of permutations generated by specific automorphism operations. We also show results using automorphisms, namely those given in Table 1. Details of these automorphisms are shown in Section 4.

We also used other programs to compute good lower bounds:

1. Programs which find a maximum size clique in a graph.
2. Programs which combine randomness with a Greedy approach.

That is, the first constructs a graph with a node for each permutation on n symbols and an edge connecting two nodes whose permutations are at Kendall-τ distance at least d. The set of nodes (permutations) in a maximum size clique in this graph is a (n, d)-PA. The second initially chooses randomly a specified size set of permutations at pairwise Kendall-τ distance d, and then proceeds through all remaining permutations in lexicographic order and adds them to the set if they have Kendall-τ distance at least d.

In Tables 1 and 2 are given sporadic results obtained by these techniques. Blank positions in our tables signify other papers have the best lower bounds known *e.g.* [5], [10]. All other lower bounds we give are larger than previous lower bounds, except for the two noted in Table 1.

n:d	3	4	5	6	7	8	9
6	$102^{(*)}$						
7	$588^{(*)}$	336	126	84	42		
8	3,752	2,240	672	448	168		
9					1,008		288

Table 1: Improved lower bounds on $P(n, d)$ by automorphisms. (The bounds for $P(6, 3)$ and $P(7, 3)$ are from [10] and [5], respectively.)

n:d	3	4	5	6	7	8	9
8						115	57
9	26,831	15,492	3,882	2,497		608	
10	233,421	133,251	29,113	18,344	5,629	3,832	1,489
11			247,014	153,260	42,013	28,008	9,747
12							73,068

n:d	10	11	12	13	14	15
7	13	8	7	4		
8	43	26	21	15	12	8
9	195	100	77	46	37	24
10	1,066	491	370	195	152	89
11	6,890	2,861	2,108	1,005	768	409
12	50,649	19,227	13,935	6,087	4,564	2,239

Table 2: Improved lower bounds by random Greedy.

In [2] Barg and Mazumdar described their Theorem 4.5, which is given below:

Theorem 1. *[2] Let $m = ((n-2)^{t+1} - 1)/(n-3)$, where $n-2$ is a prime power. Then*

$$P(n, 2t+1) \geq \frac{n!}{t(t+1)m}.$$

This was improved by Wang, Zhang, Yang, and Ge in [11].

Theorem 2. *[11] Let $m = ((n-2)^{t+1} - 1)/(n-3)$, where $n-2$ is a prime power. Then*

$$P(n, 2t+1) \geq \frac{n!}{(2t+1)m}.$$

For example, by choosing $t = 1$ and $n = 11$, one obtains, by Theorem 2, $P(11, 3) \geq 1{,}330{,}560$. Theorem 2 applies only when n is two greater than a power of a prime. To compute good lower bounds for $P(n, d)$ when n is not two greater than a power of a prime, one needs other techniques. The lower bounds given by Theorem 2 are close to corresponding upper bounds when the Kendall-τ

distance is small, but not so close when the Kendall-τ distance is close to n. Our Theorems 6 and 7, described below, give better lower bounds when the Kendall-τ distance is close to n.

The following theorem from [8] allows one to obtain good lower bounds for even Kendall-τ distances.

Theorem 3. *[8] For all $n \geq 1$ and even $d \geq 2$, we have $P(n,d) \geq \frac{1}{2}P(n,d-1)$.*

Theorem 4. *[8] For all $n, d \geq 1$ we have $P(n+1,d) \leq (n+1) \cdot P(n,d)$.*

Using Theorems 4 and 2 we have $P(14,11) \geq P(15,11)/15 \geq 15!/(11 \cdot 402234 \cdot 15) \approx 19,703.2$

Theorem 5. *[8] For all $n, d > 1$ we have $P(n+1,d) \geq \lceil \frac{n+1}{d} \rceil P(n,d)$.*

For example, to compute a lower bound for $P(14,11)$ one can use, iteratively, Theorem 5 to obtain $P(14,11) \geq \lceil \frac{14}{11} \rceil \cdot \lceil \frac{13}{11} \rceil \cdot P(12,11) = 4 \cdot P(12,11)$. By computation (using the random greedy algorithm) we have $P(12,11) \geq 19,277$, so $P(14,11) \geq 76,908$. We next give generalizations of Theorem 5 that yield improvements.

Let $S_{n,m}$ be the set of permutations on $[1...n]$ with the restriction that the first $n-m$ symbols are in sorted order, for a given $m < n$. A set $A \subseteq S_{n,m}$ with Kendall-τ distance d is called a (n,m,d)-PA or (n,m,d)-array. Let $P(n,m,d)$ denote the maximum cardinality of any (n,m,d)-array A.

Theorem 6. *For any $m < n$ and d, $P(n,d) \geq P(n,m,d) \cdot P(n-m,d)$.*

Proof. Let A be a (n,m,d)-array and B be a $(n-m,d)$-array. For each permutation π in A and each permutation τ in B, form the permutation (π,τ) by substituting the $n-m$ symbols in the order given by τ for the first $n-m$ symbols, given in order, in π.

It is easily seen that $d((\pi,\tau),(\rho,\sigma)) \geq d$, if either $\pi \neq \rho$ or $\sigma \neq \tau$. That is, for $\pi, \rho \in A$, if $\pi \neq \rho$, then $d(\pi,\rho) \geq d$. Clearly, changing the order of the other $n-m$ symbols, which appear in order in permutations in A, does not make the distance smaller. A symmetric argument applies when σ, τ are different permutations in the $(n-m,d)$-array B. $\square$

In [8] Theorem 5 was proved using the set $\{1, d+1, 2d+1, \ldots, \lceil \frac{n+1}{d} \rceil d+1\}$, which corresponds to a $(n+1,1,d)$-array. In general, a (n,m,d)-array can be much larger than one obtained by the iterative use of Theorem 5. For example, for all n, we give $(n,2,3)$-arrays with $\frac{n(n+1)}{6}$ permutations, when $n-1$ is not divisible by 3. Also, for $n = 14$ we computed a $(14,2,11)$-array with 5 permutations $\tau_1, \ldots, \tau_5$ shown in Table 3. Thus, using Theorem 6 we obtain $P(14,11) \geq 5 \cdot P(12,11) \geq 5 \cdot 19,277 = 96,135$ which is a better lower bound than obtained by Theorem 5.

One can also improve on Theorem 6. For each permutation, say τ in a (n,m,d)-array A, one can generally find a larger set of permutations than in the best $(n-m,d)$-array. Let $P_\tau(n,d)$ denote the maximum cardinality of any (n,d) PA with the highest m symbols in the same positions as in τ, but where the other $n-m$ symbols can be in any order. We also denote it by $P(n,d;i_1,\ldots,i_m)$, where $i_1,\ldots,i_m$ are the fixed positions of symbols $n-m+1,\ldots,n$, not necessarily in that order.

Theorem 7. *For any (n,m,d)-array A, $P(n,d) \geq \sum_{\tau \in A} P_\tau(n,d)$.*

Proof. Let A be a (n,m,d)-array and, for each permutation $\pi \in A$, let τ be a permutation in an (n,d)-PA with the highest m symbols in the same position as in π. Form the new permutation (π,τ) by substituting the $n-m$ symbols in the order given by τ for the first $n-m$ symbols, given in order, in π.

It is easily seen, as in the proof of Theorem 6, that $d((\pi,\tau),(\rho,\sigma)) \geq d$, if either $\pi \neq \rho$ or $\sigma \neq \tau$. $\square$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$P_{\tau_i}(14,11)$
τ_1	0	0	0	0	0	0	13	14	0	0	0	0	0	0	47,851
τ_2	0	0	14	0	0	0	0	0	0	0	0	0	0	13	36,250
τ_3	0	13	0	0	0	0	0	0	0	0	0	0	0	14	19,227
τ_4	13	14	0	0	0	0	0	0	0	0	0	0	0	0	19,227
τ_5	0	0	0	0	0	0	0	0	0	0	0	0	14	13	19,227

Table 3: $(14,2,11)$-array with 5 permutations $\tau_1,\ldots,\tau_5$. Since the first 12 symbols in all τ_i are sorted, they are replaced by zeros. The last column contains lower bounds for $P_{\tau_i}(14,11), i=1,\ldots,5$.

For example, we saw the result $P(14,11) \geq 96,125$ using Theorem 6, with a $(14,2,11)$-array with five permutations $\tau_i, i=1,\ldots,5$. We computed lower bounds for $P_{\tau_i}(14,11)$, see the last column in Table 3. By Theorem 7, we obtain the improved lower bound of $P(14,11) \geq \sum_{i=1}^{5} P_{\tau_i}(14,11) \geq 141,782$.

2 Bounds for $P(n,m,d)$

There are $\frac{n!}{(n-m)!}$ permutations in $S_{n,m}$ for finding $P(n,m,d)$. When m is small, this is relatively small compared to the $n!$ permutations to explore for finding $P(n,d)$. Also, $P(n,m,d)$ generalizes $P(n,d)$ as $P(n,d) = P(n,n,d)$. Finding exact values or bounds for $P(n,m,d)$ is an interesting problem in its own right. Clearly, $P(n,1,d) = \lceil n/d \rceil$. In general, by Theorem 5

$$P(n,m,d) \geq \left\lceil \frac{n}{d} \right\rceil \cdot \left\lceil \frac{n-1}{d} \right\rceil \cdot \cdots \cdot \left\lceil \frac{n-m+1}{d} \right\rceil . \tag{1}$$

We denote by ε the identity permutation $(1,2,\ldots,n)$.

Proposition 8. *$P(n,m,d) \geq 2$ if $d \leq mn - m(m+1)/2$. The bound for d is tight for all $n > m \geq 1$.*

Proof. Let $\pi = (n, n-1, \ldots, n-m+1, 1, 2 \ldots, n-m)$. The bubble sort for π uses $n-1$ transpositions for symbol n, $n-2$ transpositions for symbol $n-1$, etc. Then $d(\varepsilon,\pi) = (n-1)+(n-2)+\cdots+(n-m) = nm - (1+2+\cdots+m) = mn - m(m+1)/2$.

The bound is tight since for any permutation $\sigma \neq \pi$, $d(\varepsilon,\sigma) < mn - m(m+1)/2$. ☐

We improve the bound in Equation 1 for $m = 2$.

Theorem 9. *For any $d \geq 1$,*
(a) $P(n,2,d) \geq 3$ if $d \leq n + \lfloor n/3 \rfloor - 2$.
(b) $P(n,2,d) \geq 5$ if $d \leq n-2$.

Proof. (a) Let $\tau_1 = (n-1, n, 1, 2, \ldots, n-2), \tau_2 = (1, \ldots, x-1, n-1, x, \ldots, n-2, n)$ and $\tau_3 = (1, \ldots, x, n, x+1, \ldots, n-1)$ where $x = \lfloor n/3 \rfloor$, see an example in Table 4. Transformation of τ_1 to τ_2 requires $n-1$ transpositions for symbol $n-1$ and $x-1$ transpositions for symbol n. Then $d(\tau_1,\tau_2) = n+x-2 \geq d$. Similarly $d(\tau_1,\tau_3) = (n-2)+x \geq d$, and $d(\tau_2,\tau_3) = (n-x)+(n-x-2) = 2n-2x-2 \geq n+x-2 \geq d$.

	1	2	3	4	5	6	7	8	9
τ_1	8	9	1	2	3	4	5	6	7
τ_2	1	2	9	3	4	5	6	7	8
τ_3	1	2	3	8	4	5	6	7	9

Table 4: $P(9, 2, 10) \geq 3$.

(b) Suppose $n = 2k$. Consider 5 permutations $\tau_i, i = 1, \ldots, 5$ where symbols $n-1$ and n are placed at positions 1 and 2 for τ_1, $n-1$ and n for τ_2, k and $k+1$ for τ_3, 1 and n for τ_4, n and 1 for τ_5, see an example in Table 5. We show that $d(\tau_i, \tau_j) \geq n-2$ if $1 \leq i < j \leq 5$. For all pairs $i, j \in \{1, 2, 4, 5\}$ with $i < j$, transformation of τ_i to τ_j requires $n-2$ transpositions for only one of two symbols $n-1$ or n. Transformation of τ_3 to any $\tau_i, i = 1, 2, 4$ requires $k-1$ transpositions for each symbol $n-1$ and n. Transformation of τ_3 to any τ_5 requires $k-1$ transpositions for each symbol $n-1$ and n after transposition of $n-1$ and n.

	1	2	3	4	5	6	7	8	9	10	11	12
τ_1	11	12	0	0	0	0	0	0	0	0	0	0
τ_2	0	0	0	0	0	0	0	0	0	0	11	12
τ_3	0	0	0	0	0	11	12	0	0	0	0	0
τ_4	11	0	0	0	0	0	0	0	0	0	0	12
τ_5	12	0	0	0	0	0	0	0	0	0	0	11

Table 5: $P(12, 2, 10) \geq 5$. The first 10 symbols in all τ_i are in the sorted order and replaced by zeros.

	1	2	3	4	5	6	7	8	9	10	11	12	13
τ_1	12	13	0	0	0	0	0	0	0	0	0	0	0
τ_2	0	0	0	0	0	0	0	0	0	0	0	12	13
τ_3	0	0	0	0	0	13	12	0	0	0	0	0	0
τ_4	12	0	0	0	0	0	0	0	0	0	0	0	13
τ_5	13	0	0	0	0	0	0	0	0	0	0	0	12

Table 6: An example for $P(13, 2, 11) \geq 5$.

Similarly, a $(n, 2, n-2)$-array can be constructed for $n = 2k+1$ where symbols n and $n-1$ are placed at positions k and $k+1$ for τ_3, see an example in Table 6. $\square$

We have constructed a program for computing $P(n, m, d)$ for various values of n, m, and d. For each of the $\binom{n}{m}$ positions for m symbols out of n, and each of the possible $m!$ orders of the m symbols, the program uses the random/Greedy strategy described earlier. That is, it chooses a specified number of random choices first and then tries adding all remaining possible permutations in increasing order. When m is small, the program finds solutions quickly. It allows one to compute $P(15, 12)$, for example, without examining all 15! permutations of 15 symbols. That is, by Theorem 6 one can first compute, for example, $P(15, 3, 12)$, which as shown in Table 9 is at least 12, and then compute $P(12, 12)$.

As shown in Table 11 these are useful for obtaining improved lower bounds for $P(n, d)$ when the Kendall-τ distance d is close to n. We give lower bounds for $P(n, m, d)$, for $8 \le d \le 15$ and $10 \le n \le 20$ in Tables 7, 8, 9, and 10.

n:m	2	3	4	5	6
10	5	14	37	113	335
11	5	16	55	186	645
12	6	21	73	285	1145
13	6	26	99	428	1920
14	8	31	130	625	3117
15	8	37	172	884	4872
16	10	45	219	1233	7367
17	10	52	278	1676	10828
18	13	61	344	2227	15567
19	13	71	426	2939	21862
20	15	80	517	3805	30196

n:m	2	3	4	5	6
10	3	9	24	63	162
11	5	15	34	99	301
12	5	16	46	149	523
13	6	18	59	219	861
14	6	22	78	315	1383
15	7	26	100	445	2119
16	8	31	128	610	3165
17	8	36	162	824	4613
18	10	42	201	1097	6589
19	10	49	244	1427	9179
20	12	55	292	1827	12581

Table 7: Lower bounds for $P(n, m, 8)$ (left) and $P(n, m, 9)$ (right).

$n : m$	2	3	4	5	6
10	3	7	19	48	125
11	5	10	27	76	226
12	5	13	37	116	394
13	6	16	50	167	644
14	6	18	64	241	1011
15	6	21	83	342	1570
16	6	25	103	467	2337
17	8	30	129	629	2239
18	8	35	158	829	3185
19	10	40	192	1084	4405
20	10	46	233	4184	6017

$n : m$	2	3	4	5	6
10	3	6	13	27	73
11	3	7	16	41	128
12	3	10	22	61	214
13	5	11	31	96	344
14	5	13	37	120	539
15	5	17	55	163	810
16	6	20	70	220	1193
17	6	23	86	366	1716
18	7	26	106	472	2413
19	8	31	127	618	3362
20	8	35	151	789	4571

Table 8: Lower bounds for $P(n, m, 10)$ (left) and $P(n, m, 11)$ (right).

3 Improved Lower Bounds by Theorems 5, 6, and 7.

Each of the improved lower bounds given in Table 11 is explained in this section. Many of the computations described took weeks on Apple MacBook Air computers with an M1 or M2 processor.

- By Theorem 7, $P(12, 5) \ge P(12, 5; 2) + P(12, 5; 7) + P(12, 5; 12) \ge 318,641 + 334,200 + 246,968 = 899,809$.
- By Theorem 7, $P(12, 7) \ge P(12, 7; 3) + P(12, 7; 10) \ge 2 \cdot 64,649 = 129,298$.
- By Theorem 7, $P(12, 8) \ge P(12, 8; 3) + P(12, 8; 11) \ge 44,042 + 41049 = 85,091$.
- By Theorem 7, $P(13, 9) \ge P(13, 9; 3) + P(13, 9; 12) \ge 124,047 + 112,717 = 236,764$

$n:m$	2	3	4	5	6
10	2	6	13	26	58
11	3	7	17	40	101
12	3	9	23	59	168
13	3	10	30	84	273
14	5	13	37	117	420
15	5	16	45	159	622
16	5	17	58	216	919
17	6	20	72	287	1323
18	6	22	87	375	1859
19	6	25	103	485	2580
20	8	30	125	620	3503

$n:m$	2	3	4	5	6
10	2	4	10	20	37
11	2	6	13	28	63
12	3	7	16	40	103
13	3	9	22	56	163
14	3	10	27	79	247
15	5	12	35	106	370
16	5	15	44	141	533
17	5	16	52	181	757
18	6	18	63	242	1058
19	6	20	73	308	1447
20	6	23	90	390	1965

Table 9: Lower bounds for $P(n,m,12)$ (left) and $P(n,m,13)$ (right).

$n:m$	2	3	4	5	6
10	2	4	10	16	30
11	2	4	11	23	51
12	3	6	15	34	85
13	3	7	18	48	133
14	3	9	24	65	203
15	3	10	30	88	298
16	5	13	38	118	431
17	5	15	46	153	609
18	5	16	54	197	844
19	6	18	63	254	1163
20	6	20	75	323	1568

$n:m$	2	3	4	5	6
10	2	4	6	12	19
11	2	4	10	20	31
12	2	5	12	21	48
13	3	6	15	30	72
14	3	7	16	40	107
15	3	9	23	52	154
16	3	10	29	84	221
17	5	12	35	109	385
18	5	14	41	138	530
19	5	16	41	174	720
20	5	17	46	220	961

Table 10: Lower bounds for $P(n,m,14)$ (left) and $P(n,m,15)$ (right).

- By Theorem 7, $P(14,9) \geq P(14,9;2,6,8)+P(14,9;2,3,5)+P(14,9;1,5,6)+P(14,9;1,6,8)+P(14,9;4,7,8)+P(14,9;4,9,10)+P(12,9;8)+P(12,9;9)+P(13,9;4,5)+P(12,9;3)+P(14,9;3,5,7)+P(14,9;3,5,14)+P(14,9;2,4,14)+P(14,9;2,7,9)+P(13,9;4,5)+P(12,9;7)+2*P(12,9;4)+P(13,9;3,5)+P(13,9;2,3)+P(12,9;3)+P(13,9;3,4)+ \geq 51,871+26,347+19,878+31,130+39,622+42,132+18,649+18,397 = 19,914+17,294+48,029+28,367+25,367+52,958+19,915+18,807+36,794+28,348+16,073+17,294+18,542 = 575,728$
- By Theorem 7, $P(13,10) \geq P(13,10;2)+P(13,10;12) \geq 2*79,104 = 158,208$.
- By Theorem 7, $P(14,10) \geq P(14,10;5,14)+P(14,10;8,14)+P(14,10;6,7)+P(14,10;1,11)+P(14,10;1,12)+P(14,10;1,2) \geq 94,643+95,052+102,965+93,157+89,021+50,649 \geq 525,427$
- By Theorem 7, $P(13,11) \geq P(13,11;2)+P(13,11;13) \geq 31,809+19,227 = 51,046$.
- By Theorem 7, $P(14,11) \geq P(14,11;7,8)+P(14,11;14,3)+P(14,11;13,14)+P(14,11;1,2)+P(14,11;1,14) \geq 47,851+36,250+3*19,227 = 141,782$.
- By Theorem 7, $P(15,11) \geq P(15,11;1,7,9)+P(15,11;9,10,15)+P(15,11;11,14,15)+$

n:d	5	7	8	9	10	11	12	13	14	15
12	899,809	129,298	85,091							
13				236,764	158,208	51,046	29,859	14,158	10,756	5,527
14				595,728	525,427	141,782	100,813	52,565	41,673	15,674
15						1,049,633	524,817	105,130	83,346	37,104
16						2,099,266	1,049,634	267,828	173,432	74,208
17										244,051

Table 11: Improved lower bounds using Theorems 5, 6 and 7. Blanks indicate other methods have the best lower bounds known *e.g.* [11] or, for n=12, the best lower bounds are in Table 2.

$P(15,11;8,9,11)+P(15,11;6,10,15)+P(15,11;5,7,13)+P(15,11;5,6,15)+P(15,11;4,12,14)+P(15,11;4,5,6)+P(15,11;3,14,15)+P(15,11;2,7,11)+P(15,11;1,13,15)+P(15,11;1,2,3)+P(15,11;1,2,15)+P(15,11;1,2,15)+P(15,11;1,2,13)+P(15,11;1,9,11) \geq 70,509+47,069+36,430+93,986+85,010+138,475+47,027+107,707+45,837+145,804+3*19227+31,861+69,377 \geq 1,049,633$

- By Theorem 4, $P(16,11) \geq 2*P(15,11) \geq 2*1,049,633 = 2,099,266$
- By Theorem 7, $P(13,12) \geq P(13,12;7) \geq 29,859$.
- By Theorem 7, $P(14,12) \geq P(14,12;7,8)+P(14,12;13,14)+P(14,12;14,2)+P(14,12;1,14)+P(14,12;1,2) \geq 35,709+13,935+23,299+19,227+19,227 = 100,813$.
- By Theorem 3, $P(15,12) \geq \frac{1}{2}P(15,11) \geq 524,817$.
- By Theorem 4, $P(16,12) \geq 2*P(15,12) \geq 1,049,634$
- By Theorem 7, $P(13,13) \geq P(13,13;7) \geq 14,158$.
- By Theorem 7 $P(14,13) \geq P(14,13;7,13) + P(14,13;6,14) + P(14,13;3,4) \geq 23,388 + 14,073 + 15,104 \geq 52,565$.
- By Theorem 5, $P(15,13) \geq 2*P(14,13) \geq 2*52,565 = 105,130$.
- By Theorem 6, $P(16,13) \geq P(12,13)*P(16,4,13) \geq 6,087*44 = 267,828$.
- By Theorem 7, $P(13,14) \geq P(13,14;7) \geq 10,756$.
- By Theorem 7, $P(14,14) \geq P(14,14;1,3) + P(14,14;4,14) + P(14,14;6,11) \geq 8,036 + 10,060 + 23,577 = 41,673$
- By Theorem 5, $P(15,14) \geq 2*P(14,14) \geq 2*41,673 = 83,346$.
- By Theorem 6, $P(16,14) \geq P(12,14)*P(16,4,14) \geq 4,564*38 = 173,432$.
- By Theorem 7, $P(13,15) \geq P(13,15;7) \geq 5,527$.
- By Theorem 7, $P(14,15) \geq P(14,15;6,14) + P(14,15;14,6) + P(14,15;2,3) \geq 5,493 + 5,493 + 4,688 = 15,674$.
- By Theorem 7, $P(15,15) \geq P(15,15;3,4,7,8) + 3*P(11,15) + P(15,15;4,5,6,7) + 3*P(14,15;6,7,8) + P(13,15:2,10) + P(15,15;2,3,4,13) + P(3,5,7,11) + P(15,15,2,4,10,11) + 2P*(13,15;2,3) + P(12,15;3) + 4*P(12,15;2) + P(14,15;3,4,5) \geq 4,279 + 3*409 + 1,787 + 3*1,848 + 1,738 + 1,964 + 7,798 + 5,773 + 2*879 + 895 + 4*743 + 1,369 \geq 37,104$.

- By Theorem 5, $P(16, 15) \geq 2 * P(16, 15) \geq 74,208$.
- By Theorem 6, $P(17, 15) \geq P(12, 15) * P(17, 5, 15) \geq 2,239 * 109 = 244,051$.

4 Automorphism Lower Bounds

It is known that for a permutation $\pi(x) : \mathbb{F}_q \to \mathbb{F}_q$, where $\mathbb{F}_q$ denotes a finite field of order q, the operations of multiplying by a non-zero constant a, adding a constant c, and adding to the argument a constant b, each yield another permutation on $\mathbb{F}_q$. That is, $a\pi(x + b) + c$, for all non-zero a and all $b, c \in \mathbb{F}_q$, is again a permutation. We use this to search for sets of permutations at specified Kendall-τ distance d. That is, the search can be done for a set of representative permutations and expanded into a full set of permutations using operations on the representatives. Our program verifies that the full set of permutations has the stipulated Kendall-τ distance.

Example. Use the operation $\pi(x) + c$ on the following 17 representatives. This gives 102 permutations for $P(6, 3)$.

0 1 2 3 5 4	0 1 2 4 5 3	0 1 3 5 4 2	0 1 5 4 2 3	0 2 3 4 1 5	0 2 4 5 1 3
0 2 5 3 4 1	0 3 1 4 2 5	0 3 2 5 1 4	0 3 4 2 5 1	0 3 5 4 1 2	0 4 1 5 3 2
0 4 2 1 3 5	0 4 5 3 2 1	0 5 2 1 3 4	0 5 3 1 2 4	0 5 4 2 1 3	

Example. Use the operations $a\pi(x) + c$ on the following 14 representatives. This gives 1,008 permutations for $P(9, 7)$.

0 1 2 4 8 3 7 5 6	0 1 2 7 8 5 3 4 6	0 1 3 4 7 2 8 6 5	0 1 3 8 2 6 7 4 5
0 1 3 8 4 6 5 7 2	0 1 4 5 6 7 3 8 2	0 1 4 5 8 2 7 6 3	0 1 6 2 3 4 7 8 5
0 1 6 2 8 7 5 4 3	0 1 6 4 5 2 3 8 7	0 1 6 7 3 4 8 5 2	0 1 7 2 4 6 8 5 3
0 1 7 4 8 3 5 2 6	0 1 8 5 7 4 6 3 2		

Example. Use the operations $a\pi(x) + c$ on the following 8 representatives. This gives 576 permutations for $P(9, 8)$.

0 1 2 3 8 4 6 5 7	0 1 2 5 8 6 3 7 4	0 1 4 5 2 8 6 7 3	0 1 5 3 2 4 6 8 7
0 1 5 6 4 8 3 7 2	0 1 6 4 7 2 5 8 3	0 1 6 7 3 2 8 5 4	0 1 8 3 6 5 7 2 4

Example. Use the operations $a\pi(x) + c$ on the following four representatives. This gives 288 permutations for $P(9, 9)$.

0 1 2 6 5 8 7 4 3	0 1 3 8 4 5 2 6 7	0 1 4 6 5 3 7 2 8	0 1 5 2 4 7 3 6 8

Example. Use the operations $\pi(x) + c$ on the following 12 representatives. This gives 84 permutations for $P(7, 6)$.

0 1 3 6 5 4 2	0 1 4 2 3 6 5	0 1 6 2 5 4 3	0 2 3 4 1 5 6	0 2 3 6 5 1 4
0 3 4 6 1 2 5	0 3 5 4 1 2 6	0 4 5 6 3 1 2	0 5 2 4 3 6 1	0 5 3 6 1 2 4
0 6 3 5 4 2 1	0 6 4 2 1 3 5			

Example. Use the operation $a\pi(x) + c$ on 8 permutations. This gives 448 permutations for $P(8, 6)$.

0 1 7 4 5 6 2 3	0 2 1 5 3 4 6 7	0 2 6 4 7 3 1 5	0 3 7 5 4 2 1 6
0 5 4 6 7 1 2 3	0 7 3 1 2 6 5 4	0 7 5 4 3 6 1 2	0 7 6 4 2 1 3 5

Example. Use the operation $a\pi(x) + c$ on 67 permutation representatives. This gives 3,752 permutations for $P(8, 3)$.

0 1 2 3 4 5 6 7	0 1 2 5 3 6 7 4	0 1 3 5 7 2 6 4	0 1 5 4 3 6 2 7	0 1 6 2 7 3 4 5
0 1 6 3 4 2 7 5	0 1 6 7 4 5 2 3	0 1 7 3 2 5 6 4	0 1 7 5 3 2 4 6	0 1 7 5 6 3 4 2
0 2 3 5 1 4 7 6	0 2 3 5 7 6 4 1	0 2 3 6 5 4 7 1	0 2 4 1 6 5 7 3	0 2 4 5 6 3 1 7
0 2 5 1 7 4 3 6	0 2 5 3 4 6 7 1	0 2 5 4 3 1 6 7	0 2 5 6 4 1 7 3	0 2 6 4 3 5 1 7
0 2 6 4 7 1 5 3	0 3 1 5 4 7 2 6	0 3 2 4 1 7 6 5	0 3 2 5 4 7 1 6	0 3 2 6 1 4 5 7
0 3 6 2 4 5 1 7	0 3 7 4 5 6 2 1	0 3 7 5 4 2 1 6	0 4 1 6 2 3 5 7	0 4 2 7 3 1 5 6
0 4 2 7 5 6 1 3	0 4 5 6 2 1 3 7	0 4 6 1 7 2 3 5	0 4 6 2 5 3 7 1	0 4 6 2 7 1 5 3
0 4 7 5 2 3 1 6	0 4 7 6 3 5 2 1	0 5 1 6 7 4 3 2	0 5 1 7 3 6 2 4	0 5 2 1 6 3 7 4
0 5 2 3 6 4 1 7	0 5 2 6 4 3 7 1	0 5 3 1 4 6 2 7	0 5 3 2 6 1 7 4	0 5 3 4 1 2 7 6
0 5 3 7 6 1 4 2	0 5 4 6 2 7 1 3	0 5 4 6 3 1 2 7	0 5 6 3 1 2 7 4	0 5 6 3 7 4 1 2
0 5 7 6 4 3 1 2	0 6 1 5 2 3 4 7	0 6 2 4 3 7 5 1	0 6 3 1 7 4 5 2	0 6 3 7 2 4 5 1
0 6 4 3 5 7 1 2	0 6 5 1 7 3 2 4	0 6 7 1 3 5 4 2	0 6 7 5 3 2 1 4	0 7 1 2 3 4 5 6
0 7 1 3 5 4 6 2	0 7 1 4 3 6 2 5	0 7 3 4 2 1 5 6	0 7 3 6 1 4 2 5	0 7 4 6 3 1 2 5
0 7 4 6 5 2 3 1	0 7 5 1 2 3 6 4			

Example. Use the operation $a\pi(x) + c$ on 12 permutation representatives. This gives 672 permutations for $P(8, 5)$.

0 2 3 6 5 4 7 1	0 2 4 3 1 5 6 7	0 3 2 1 6 4 7 5	0 3 5 1 6 2 7 4	0 5 7 2 4 6 1 3
0 6 3 4 5 2 1 7	0 6 3 7 1 5 2 4	0 6 5 4 7 3 1 2	0 7 1 5 4 6 2 3	0 7 3 6 4 2 1 5
0 7 4 1 2 6 5 3	0 7 5 6 4 1 3 2			

Example. Use the operation $a\pi(x) + c$ on 40 permutation representatives. This gives 2,242 permutations for $P(8, 4)$.

0 1 4 5 7 6 3 2	0 1 7 3 2 5 6 4	0 2 1 3 7 4 5 6	0 2 1 5 7 4 6 3	0 2 1 6 7 5 4 3
0 2 3 6 1 5 4 7	0 2 4 3 5 6 1 7	0 2 5 3 7 4 6 1	0 2 7 1 4 5 3 6	0 2 7 3 1 4 6 5
0 2 7 3 6 5 1 4	0 2 7 6 1 4 5 3	0 3 2 1 5 7 4 6	0 3 5 6 4 7 1 2	0 3 5 7 6 1 2 4
0 3 6 2 5 1 7 4	0 4 1 6 2 3 5 7	0 4 1 7 6 2 3 5	0 4 2 1 5 6 3 7	0 4 2 5 7 6 3 1
0 4 2 7 1 5 6 3	0 4 3 1 7 5 6 2	0 4 3 5 6 1 7 2	0 5 2 1 6 3 7 4	0 5 3 2 6 1 7 4
0 5 3 2 7 1 4 6	0 5 4 2 1 3 6 7	0 5 4 7 6 2 3 1	0 5 6 2 1 7 4 3	0 5 6 4 1 3 2 7
0 5 7 1 6 4 2 3	0 5 7 3 2 4 6 1	0 6 1 2 4 3 5 7	0 6 7 2 4 3 1 5	0 7 1 2 6 3 5 4
0 7 2 5 1 4 6 3	0 7 2 5 3 6 4 1	0 7 4 3 1 5 2 6	0 7 5 1 4 2 3 6	0 7 5 6 2 1 4 3

Example. Use the operation $a\pi(x) + c$ on 3 permutation representatives. This gives 168 permutations for $P(8, 7)$.

0 5 3 1 4 6 2 7	0 6 1 3 2 5 7 4	0 7 3 1 2 6 5 4

Example. Use the operation $\pi(x) + c$ on the following 48 permutations. This gives 336 permutations for $P(7, 4)$.

0 1 2 4 3 6 5	0 1 2 5 4 6 3	0 1 3 2 6 5 4	0 1 3 4 2 5 6	0 1 4 5 6 2 3
0 1 5 3 6 2 4	0 1 6 2 3 4 5	0 1 6 5 2 4 3	0 2 1 3 5 4 6	0 2 3 4 1 6 5
0 2 3 6 5 4 1	0 2 4 1 5 3 6	0 2 4 6 5 1 3	0 2 5 1 6 3 4	0 2 5 3 4 1 6
0 2 5 6 4 3 1	0 2 6 1 4 3 5	0 3 1 5 2 4 6	0 3 1 6 5 4 2	0 3 2 5 1 6 4
0 3 2 6 1 4 5	0 3 4 1 6 2 5	0 3 4 2 5 1 6	0 3 4 5 6 1 2	0 3 6 5 4 2 1
0 4 2 1 6 3 5	0 4 2 5 3 6 1	0 4 3 1 5 2 6	0 4 3 6 2 5 1	0 4 5 1 2 3 6
0 4 6 1 3 2 5	0 4 6 5 1 2 3	0 5 1 2 3 4 6	0 5 2 4 1 6 3	0 5 3 1 4 6 2
0 5 3 6 2 1 4	0 5 4 1 6 3 2	0 5 4 3 6 2 1	0 5 6 1 4 2 3	0 6 1 4 2 5 3
0 6 1 5 3 4 2	0 6 2 5 3 1 4	0 6 3 1 4 2 5	0 6 3 2 4 5 1	0 6 3 5 1 2 4
0 6 4 2 3 1 5	0 6 4 5 3 2 1	0 6 5 2 4 1 3		

Example. Use the operation $\pi(x) + c$ on the following 18 permutations. This gives 126 permutations for $P(7, 5)$.

0 1 4 2 5 3 6	0 1 4 6 3 2 5	0 1 5 2 6 4 3	0 2 1 3 5 4 6	0 2 4 5 6 3 1
0 2 6 4 1 3 5	0 3 1 5 6 4 2	0 3 2 4 5 1 6	0 3 2 6 1 4 5	0 3 5 4 6 2 1
0 4 3 1 5 2 6	0 4 3 6 2 1 5	0 4 5 1 6 3 2	0 5 1 3 4 2 6	0 5 3 2 6 1 4
0 6 1 2 5 3 4	0 6 5 2 4 1 3	0 6 5 3 4 1 2		

5 Patterns for $P(n, m, d)$

In this section, let us, for convenience, describe general patterns for strings (permutations) in $P(n, 2, d)$ and $P(n, 3, d)$, by replacing the symbols $[1 \ldots n-2]$ ($[1 \ldots n-3]$, respectively), which are in order, by blank symbols, *i.e.* '-'.

For example, for $P(5, 2, 3)$, we have the set

{ 4 5 - - - , - 5 4 - - , - - 4 5 - , - - - 5 4 , 4 - - - 5 , 5 - - - 4 }.

It is easy to verify that the Kendall-τ distance between any two strings in this set is at least 3. This set agrees with that found by our program, namely $P(5, 2, 3) \geq 6$.

Also, for $P(10, 2, 3)$, we have the set

{ 9 10 - - - - - - - - ,	- 10 9 - - - - - - -,	- - 9 10 - - - - - - ,	- - - 10 9 - - - - -,
- - - - 9 10 - - - -,	- - - - - 10 9 - - -,	- - - - - - 9 10 - -,	- - - - - - - 10 9 -,
- - - - - - - - 9 10,	9 - - - 10 - - - - -,	10 - - - 9 - - - - -,	- - 9 - - - 10 - - -,
- - 10 - - - 9 - - -	- - - 9 - - - - 10 -,	- - - 10 - - - - 9 -,	- - - - - 9 - - - 10,
- - - - - 10 - - - - 9,	9 - - - - - 10 - - -,	10 - - - - - 9 - - -,	- 9 - - - - - - - 10,
- 10 - - - - - - - 9 }.			

It is easy to verify that the Kendall-τ distance between any two strings in this set is at least 3. This set agrees with that found by our program, namely $P(10, 2, 3) \geq 21$.

These examples show that sets of strings that form a $(n, 2, 3)$-array contain easily recognized patterns. It is an interesting open question if such patterns can be determined for other choices of n, m, and d.

Along these lines, for $d = 3$, consider $\pi_1(a, b, c) = \ldots, n-1, \ldots, n, \ldots$ and $\pi_2(a, b, c) = \ldots, n, \ldots, n-1, \ldots$, where a, b, c denote the number of symbols in the 3 gaps represented by the "$\ldots$". We will use $\pi_1(a, b, c)$ for $a = 0, 2, 4, \ldots$ and $b = 0, 3, 6, \ldots$, and $\pi_2(a, b, c)$ for $a = 1, 3, 5, \ldots$

and $b = 0, 3, 6, \ldots$, for each choice of a and b for which the resulting string has length at most n.

Using $\pi_1(a, b, c)$ and $\pi_2(a, b, c)$, it can be observed that $P(n, 2, 3) \geq \frac{n(n+1)}{6}$, for $n \not\equiv 1 \bmod 3$ and $P(n, 2, 3) \geq \frac{(n+2)(n-1)}{6}$ for $n \equiv 1 \bmod 3$. Similarly, for Kendall-τ distance 4 and for $n = 2k+1$, use $\pi_1(a, b, c)$ for $a = 0, 2, 4, \ldots$ and $b = 0, 4, 8, \ldots$; $\pi_2(a, b, c)$ for $a = 0, 2, 4, \ldots$ and $b = 3, 7, 11, \ldots$. Using these patterns, it can be observed that $P(4k+1, 2, 4) \geq 2k^2+k$ for $k \geq 1$ and $P(4k+3, 2, 4) \geq 2k^2 + 3k + 1$ for $k \geq 0$.

6 Conclusions and Open Questions

Theorems 6 and 7 improve many lower bounds. All of the bounds shown in Tables 1, 2, and 11 are improvements on previous results. The techniques described can be used to obtain other improvements, with sufficient time. Many of our computations required weeks.

Our work on good patterns for (n, m, d)-arrays is continuing. We conjecture that (n, m, d)-arrays can be used to compute improved lower bounds for $P(n, d)$, for all n, and for d close to n.

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