Digital Signal Processing

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References


• Digital Signal Processing with Computer Applications, P. A. Lynn, W. Fuerst

• Schaum’s Outline of Digital Signal Processing, 2nd Edition (Schaum's Outline Series) M. Hayes
Grading

• Assignment  %10
• Project      %20
• Midterm Exam %30
• Final Exam   %40
Course Outlines

• Introduction, Applications, History
• Discrete-Time signals and Systems
• Z Transform
• Sampling
• Linear time-invariant system
• Filter design
• Discrete and Fast Fourier Transform
Introduction

• Since the invention of calculus in the 17th century, scientists and engineers have developed models to represent physical phenomena in terms of functions of continuous variables and differential equations.

• Mathematicians of the 18th century, such as Euler, Bernoulli, and Lagrange, developed methods for numerical integration and interpolation of functions of a continuous variable.
Introduction

• Prior to 1960s: Analog signal processing technology
• The digital computers and microprocessors and developments of algorithm such as the fast Fourier transform (FFT) caused a major shift to digital technologies.
• One of the first uses of DSP was in oil prospecting, where seismic data was recorded on magnetic tape for later processing.
• Cooley and Tukey (1965) discovered an efficient algorithm for computation of Fourier transforms (FFT).
• The invention of the microprocessor paved the way for low-cost implementations of discrete-time signal processing systems.
• 1980s: IC technology improves DSP, very fast fixed-point and floating-point microcomputers were implemented
DSP Application

• Image and Video Applications
  – DVD, JPEG, Movie special effects, video conferencing,…

• Medical
  – MRI (Magnetic Resonance Imaging), Tomography, Electrocardiogram,…

• Military
  – Radar, Sonar, Space photographs, remote sensing,…

• Mechanical
  – Motor control, process control, oil and mineral prospecting,…
Discrete-time signals and systems

• Signal: something that conveys information generally about the state or behavior of a physical system.

• Mathematically represented as a function of independent variables such as
  – Time: speech signal
  – Position: digital image
  – Time and position: video
Signals Classification

- Continuous-time signal
  - Continues-time continues amplitude: analog signal
    - Example: Speech signal
  
  - Continues-time discrete amplitude
    - Example: Traffic light

- Discreet-time signal
  - Digital Signal: discrete amplitude => digital signal
    - Example: digital image

  - Discrete signal: continuous amplitude
    - Example: samples of analog signal, average monthly temperature
Discrete-time signals

• Signal-processing systems
  – Continuous-time systems: both the input and the output are continuous-time
  – Discrete-time systems: both the input and the output are discrete-time signals.
  – Digital system: both the input and the output are digital signals.

• Discrete-time signals can be generated by sampling a continuous-time signal, or directly by some discrete-time process
Digital vs Analog

• Pros
  – Noise performance
  – Flexibility, using a general computer
  – Stability/duplicability
  – Digital storage, random access

• Cons
  – Limitations of A/D & D/A
  – Power consumption
Discrete-Time Signals 
Sequences

- A Discrete-time signal is a sequences of numbers:
  \[ x = \{ x[n] \} \quad -\infty < n < \infty \]

- Periodic sampling of an analog signal
  \[ x[n] = x_a(nT) \quad -\infty < n < \infty \]
  T is the sampling period
  \[ f = 1/T \] sampling frequency

\[ x[0] \]
\[ x[-1] \]
\[ x[1] \]
\[ x[2] \]
Discrete-Time Signals

- Filter is used for noise removal and anti-aliasing
- There is also a low pass filter after D/A
Sequence Operations

- **Addition**: $y[n] = x[n] + w[n]$
- **Multiplication**: $y[n] = A \cdot x[n]$
- **Shift (delay)**: $y[n] = x[n - 1]$
- **Product (modulation)**: $y[n] = x[n] \cdot w[n]$
Basic Sequences

• Unit impulse \( \delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases} \)

  - Plays the same role as delta function
  - Its definition is simple and precise
  - Any sequence can be expressed as: \( x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \)

\[
p[n] = a_{-3}\delta[n + 3] + a_1\delta[n - 1] + a_2\delta[n - 2] + a_7\delta[n - 7]
\]
Basic Sequences

• Unit step

\[ u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \]

\[ u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \ldots \]

\[ = \sum_{k=0}^{\infty} \delta[n-k] \]

\[ u[n] = \sum_{k=-\infty}^{n} \delta[k] \]

\[ \delta[n] = u[n] - u[n-1] \]
Basic Sequences

• Exponential \( x[n] = A\alpha^n \)
  
  – If \( A \) and \( \alpha \) are real numbers, then the sequence is real.
  
  – If \( 0 < \alpha < 1 \) and \( A \) is positive, then the sequence values are positive and decrease with increasing \( n \).
  
  – If \( |\alpha| > 1 \), then the sequence grows in magnitude as \( n \) increases.
  
  – If \( -1 < \alpha < 0 \), \( x[n] \) alternate in sign, but again decrease in magnitude with increasing \( n \).
Basic Sequences

• Sinusoidal \( x[n] = A \cos(\omega_0 n + \phi) \)
  – A and \( \phi \) are real constant:

\[
\begin{align*}
\text{\ldots} & \quad 0 \quad \text{\ldots} \\
\ldots & \quad 0 & \quad \text{\ldots} \\
\end{align*}
\]
Complex number

• A complex number consists of the real and imaginary part: \( x = x_{re} + j x_{im} \)

  – Magnitude \( x \): \( |x| = \sqrt{x_{re}^2 + x_{im}^2} \)

  – Phase \( x \): \( \theta = \tan^{-1}(x_{im} / x_{re}) \)

  – Polar form: \( x = |x| e^{j\theta} = |x| \cos(\theta) + j |x| \sin(\theta) \)
Basic Sequences

• Complex exponential $A\alpha^n$
  - If $\alpha$ and $A$ are complex: $A = |A|e^{j\phi}$, $\alpha = |\alpha|e^{j\omega_0}$

$$x[n] = A\alpha^n = |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n} = |A||\alpha|^n e^{j(\omega_0 n + \phi)}$$

$$= |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi)$$

- If $|\alpha| > 1$ exponentially growing envelope
- If $|\alpha| < 1$ exponentially decaying envelope
- If $|\alpha| = 1$ the sequence is referred to as a complex exponential sequence

$$x[n] = |A|e^{j(\omega_0 n + \phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi)$$
Basic Sequences

• Complex exponential sequence

\[ x[n] = |A|e^{j(\omega_0 n + \phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi) \]

– Both the real and imaginary parts are sinusoidal
– \( \omega_0 \) is the frequency and \( \phi \) is the phase
– \( n \) has no unit then unit of \( \omega_0 \) must be \textit{radian}
– To have a closer analogy with the continuous-time the unit of \( \omega_0 \) can be \textit{radians/sample} then unit of \( n \) is \textit{sample}
Basic Sequences

• Complex exponential sequence

\[ x[n] = |A|e^{j(\omega_0 n + \phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi) \]

- With the frequency of \((\omega_0 + 2\pi)\):

\[ x[n] = Ae^{j(\omega_0 + 2\pi)n} = Ae^{j\omega_0 n}e^{j2\pi n} = Ae^{j\omega_0 n} \]

- The same results are obtained with \((\omega_0 + 2\pi r) \quad r \in Z\)

\[ x[n] = A\cos[(\omega_0 + 2\pi r)n + \phi] = A\cos(\omega_0 n + \phi) \]

- Thus the frequency is only considered in interval of:

\[ 0 \leq \omega_0 \leq 2\pi \quad \text{or} \quad -\pi < \omega_0 \leq \pi \]
Basic Sequences

• Complex exponential periodicity:
  – In the continuous –time a sinusoidal signal is periodic with the period of $2\pi/f$
  – In the discrete-time :
    \[ x[n] = x[n + N] \text{ for all } n \]
    \[ A\cos(\omega_0 n + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi) \]
    \[ \omega_0 N = 2\pi k \implies N = \frac{2\pi k}{\omega_0} \text{ and } N \text{ must be integer} \]
• The periodicity depends on $\omega_0$ and it’s not necessarily periodic with period of $2\pi/\omega_0$
  Exp: $\omega_0 = \frac{3\pi}{4} \implies N = \frac{2\pi k}{3\pi/4} = 8k/3, \implies N = 8, k = 3$
  $\omega_0 = 1 \implies N = 2\pi k \implies \text{there is no integer values for } N \text{ and } k$
Frequency domain in discrete-time sequences

- Continuous-time: \( x(t) = A \cos(\Omega_0 t + \phi) \)
  as \( \Omega_0 \) increases, \( x(t) \) oscillates more rapidly.

- Discrete-time: \( x[n] = A \cos(\omega_0 n + \phi) \) as \( \omega_0 \)
  increases from 0 to \( \pi \), \( x[n] \) oscillates more rapidly.

- As \( \omega_0 \) increases from \( \pi \) to \( 2\pi \), \( x[n] \) oscillates slower.

- Frequencies around \( 2\pi \) are indistinguishable from
  frequencies around 0.

- Low frequencies refer to frequencies around \( 2\pi k \)

- High frequencies refer to \( \omega_0 \) around \( \pi + 2\pi k \)
Frequency domain in discrete-time sequences

- $\omega_0 = 0$ or $\omega_0 = 2\pi$
- $\omega_0 = \pi/8$ or $\omega_0 = 15\pi/8$
- $\omega_0 = \pi/4$ or $\omega_0 = 7\pi/4$
- $\omega_0 = \pi$
Discrete-Time System

- A transformation or operator that maps inputs into outputs sequences
  \[ y[n] = T\{ x[n] \} \]

- Example
  - Ideal delay system  \[ y[n]=x[n - n_d] \quad -\infty < n < \infty \]
  - Moving average:
    - Computes the average of \((M_1 + M_2 + 1)\) samples around the nth sample
    \[
    y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n - k]
    \]
Discrete-Time System

• Memoryless Systems
  – The output $y[n]$ at $n$ depends only on the input $x[n]$ at the same value of $n$
  – Example:
    \[ y[n] = (x[n])^2 \]

– Systems with time delay or time advance have memory and the system is not memoryless
  \[ y[n] = x[n - n_d] \]
Linear Systems

• Linear systems follow the principle of superposition:
  – Additivity property
    \[ T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n] \]
  – Homogeneity or scaling property
    \[ T\{a \cdot x[n]\} = a \cdot T\{x[n]\} = a \cdot y[n] \]
  – Combination of two properties
    \[ T\{a x_1[n] + b x_2[n]\} = a \cdot T\{x_1[n]\} + b \cdot T\{x_2[n]\} \]
  – General Form:
    \[ x[n] = \sum_k a_k x_k[n] \quad y[n] = \sum_k a_k y_k[n] \]
Linear Systems

• Example: Accumulator

\[ y[n] = \sum_{k=-\infty}^{n} x[k] \]

\[ x[n] = \alpha x_1[n] + \beta x_2[n] \]

\[ y[n] = \sum_{k=-\infty}^{n} (\alpha x_1[n] + \beta x_2[n]) \]

\[ = \sum_{k=-\infty}^{n} (\alpha x_1[n]) + \sum_{k=-\infty}^{n} \beta x_2[n] \]

\[ = \alpha \sum_{k=-\infty}^{n} x_1[n] + \beta \sum_{k=-\infty}^{n} x_2[n] \]

\[ = \alpha y_1[n] + \beta y_2[n] \quad \Rightarrow \quad \text{Linear} \]
Nonlinear System

• Example 1: \[ y[n] = x^2[n] - x[n-1] \cdot x[n+1] \]
\[
x[n] = \alpha x_1[n] + \beta x_2[n]
\]
\[
y[n] = (\alpha x_1[n] + \beta x_2[n])^2
\]
\[-(\alpha x_1[n-1] + \beta x_2[n-1]) \cdot (\alpha x_1[n+1] + \beta x_2[n+1])
\]
\[\neq \alpha y_1[n] + \beta y_2[n] \Rightarrow \text{Nonlinear} \]

• Example 2: \[ w[n] = \log_{10}(|x[n]|) \]
\[
x_1[n]=1, \ x_2[n]=10 \Rightarrow w_1[n]=0, \ w_2[n]=1
\]
\[
x_2[n]=10x_1[n] \text{ but } w_2[n] \neq 10 \ w_1[n] \Rightarrow \text{Nonlinear} \]
Time-Invariant Systems

• A time-invariant or a shift-invariant system is a system that a
time shift or delay of the input sequence causes the same shift
in the output sequence.

\[ x_1[n] = x[n - n_0] \implies y_1[n] = y[n - n_0] \]

• Accumulator as a Time-Invariant system:

\[
y[n] = \sum_{k=-\infty}^{n} x[k] \quad y[n - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]
\]

\[
y_1[n] = \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k - n_0] \quad \text{Substituting } k_1 = k - n_0 , \text{ when } -\infty < k < n \implies -\infty - n_0 < k_1 < n - n_0
\]

\[
y_1[n] = \sum_{k=-\infty}^{n-n_0} x[k_1] = y[n - n_0]
\]
The compressor system

Discards \((M - 1)\) samples out of \(M\)

\[ y[n] = x[Mn] \quad \text{for } -\infty < n < \infty \]

\[ y_1[n] = x_1[Mn] = x[Mn - n_0] \]

\[ y[n - n_0] = x[M(n - n_0)] \neq y_1[n] \quad \text{The system is not Time-Invariant} \]

Other solution finding a counterexample:

\[ x[n] = \delta[n], \quad M = 2, \quad n_0 = 1, \quad x_1[n] = \delta[n - 1] \]

\[ y[n] = \delta[2n] = \delta[n] \quad \text{but } y_1[n] = \delta[2n - 1] \]

\[ y_1[n] \neq y[n - 1] \quad \text{for } n = 0 \]
Causality

• In a causal system, the current output of the system depends only on the current and previous inputs.

• If \( x_1[n] = x_2[n] \) for \( n \leq n_0 \) then \( y_1[n] = y_2[n] \)

• Example: forward difference system:
  \[
  y[n] = x[n + 1] - x[n]
  \]
  \[
  x_1[n] = \delta[n - 1], \quad x_2[n] = 0;
  \]
  \[
  y_1[n] = \delta[n] - \delta[n - 1], \quad y_2[n] = 0
  \]
  \[
  x_1[n] = x_2[n] \quad \text{for} \quad n \leq 0 \quad \text{but} \quad y_1[n] \neq y_2[n] \quad \text{for} \quad n=0 \quad \text{Non-causal}
  \]

• Example: backward difference system
  \[
  y[n] = x[n] - x[n - 1]
  \]
  – The output depends only on the present and past values of the input. So there is no way for the output at a specific time \( y[n_0] \) to incorporate values of the input for \( n > n_0 \), the system is causal.
Stability

- A system is stable if and only if each bounded input sequence produces a bounded output sequence.

  - The input $x[n]$ is bounded, when there is $B_x$ that:
    \[ |x[n]| \leq B_x < \infty \text{ for all } n \]
  - In a stable system for every bounded input, the output must be bounded
    \[ |y[n]| \leq B_y < \infty \text{ for all } n \]
  - The properties defined in this section are properties of systems, not of the inputs to a system.
  - It might that the properties hold for some inputs, but the system does not have the property because it does not hold the property for all inputs.
Testing for stability or instability

• Consider the system of: \( y[n] = (x[n])^2 \)
  - Assume that the input \( x[n] \) is bounded such that:
    \[ |x[n]| \leq B_x \text{ for all } n : \]
  - then
    \[ |y[n]| = |x[n]|^2 \leq B_x^2 \]
  - By choosing \( B_y = B_x^2 \) it proves that \( y[n] \) is bounded

• The system \( y[n] = \log_{10}(|x[n]|) = -\infty \) for any values of the \( n \) that \( x[n] = 0 \). So the output is not bounded and the system is unstable.

• Accumulator: if \( x[n] = u[n] \Rightarrow y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0, & n < 0 \\ (n+1), & n \geq 0 \end{cases} \)
  - There is no finite choice for \( B_y \) such that \((n+1) \leq B_y < \infty\)


Linear Time-Invariant Systems

- The systems which are linear and time-invariant are LTI.
- LTI systems are an important class of systems due to convenient representations and significant applications.
- A linear system can be completely characterized by its impulse response. Let $h_k[n]$ be the response of the system to $\delta[n-k]$

$$y[n] = T\{x[n]\} = T\left\{ \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k] h_k[n]$$

- Considering the property of time invariance:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = x[n] * h[n]$$

- The response of an LTI system for any input can be obtained by the convolution sum of input with the impulse response $h[n]$
Output of LTI system

\[ x_{-2}[n] = x[-2] \delta[n + 2] \]
\[ y_{-2}[n] = x[-2] h[n + 2] \]

\[ x_0[n] = x[0] \delta[n] \]
\[ y_0[n] = x[0] h[n] \]

\[ x_3[n] = x[3] \delta[n - 3] \]
\[ y_3[n] = x[3] h[n - 3] \]

\[ x[n] = x_{-2}[n] + x_0[n] + x_3[n] \]
\[ y[n] = y_{-2}[n] + y_0[n] + y_3[n] \]
Computation of the Convolution Sum

- Calculation of $h[n-k] = h[-(k-n)]$

- Reflecting $h[k]$ about the origin to obtain $h[-k]$;

- Shifting the origin of the reflected sequence to $k = n$. 
Analytical Evaluation of the Convolution Sum

Consider a system with impulse response and input:

\[ h[n] = u[n] - u[n - N] = \begin{cases} 1, & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases} \]

\[ x[n] = a^n u[n] \]

The output at \( n \) is the sums over all \( k \) of the product \( x[k]h[n - k] \)

For \( n < 0 \), \( y[n] = 0 \) because there is no overlap between \( x[k] \) and \( h[n - k] \)
Analytical Evaluation of the Convolution Sum

• If $n \leq 0$ and $n - N + 1 \leq 0$ or $0 \leq n \leq N - 1$:

\[ x[k]h[n-k] = a^k \quad \Rightarrow \quad y[n] = \sum_{k=0}^{n} a^k \]

\[ \sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a} \quad \Rightarrow \quad y[n] = \frac{1-a^{n+1}}{1-a} \]

• For $n \geq N$ \quad $x[k]h[n-k] = a^k$  \quad $n-N+1 < k \leq n$

\[ y[n] = \sum_{k=n-N+1}^{n} a^k = \frac{a^{n-N+1} - a^{n+1}}{1-a} = a^{n-N+1} \left( \frac{1-a^N}{1-a} \right) \]
Analytical Evaluation of the Convolution Sum

\[ h[n] = u[n] - u[n-N] = \begin{cases} 
1, & 0 \leq n \leq N-1 \\
0, & \text{otherwise}
\end{cases} \]

\[ x[n] = a^n u[n] \]

\[ y[n] = \begin{cases} 
0, & n < 0 \\
\frac{1-a^{n+1}}{1-a}, & 0 \leq n \leq N-1 \\
a^{n-N+1} \left( \frac{1-a^N}{1-a} \right), & N-1 < n
\end{cases} \]
Properties of LTI Systems by the Convolution

• Commutative

Substitution of $m = n - k$

$$y[n] = \sum_{m=\infty}^{-\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n]$$

$$x[n] * h[n] = h[n] * x[n]$$

– The system output is the same if the roles of the input and impulse response are reversed.

– The impulse response of a cascade combination of linear time-invariant systems is independent of their order.

– The convolution operation also distributes over addition:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$
Properties of LTI Systems by the Convolution

- Cascade connection
  - The output of the first system is the input to the second
  - The output of the second is the input to the third, etc.
  - The output of the last system is the overall output.
  - The impulse response of two cascaded LTI systems is the convolution of the impulse responses of the two systems.
  - The overall impulse response of the system is:

\[
h[n] = h_1[n] \ast h_2[n]
\]
Properties of LTI Systems by the Convolution

• Parallel connection
  – the systems have the same input, and their outputs are summed to produce an overall output.
  – Based on the distributive property of convolution the overall systems is equivalent to a single system whose impulse response is the sum of the individual impulse responses

\[
x[n] * h_1[n] + x[n] * h_2[n] = x[n] * (h_1[n] + h_2[n])
\]
Properties of LTI Systems by the Convolution

• Stability
  – Every bounded input produces a bounded output.
  – LTI systems are stable if and only if the impulse response is absolutely summable:

  \[
  s = \sum_{k=-\infty}^{\infty} |h[k]| < \infty
  \]

  \[
  |y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|
  \]

  – If \( x[n] \) is bounded:
  – Substituting \( Bx \) for \( |x[n-k]| \):
    \[
    |x[n]| \leq B_x
    \]
  – Thus, \( y[n] \) is bounded if \( s \) is
    \[
    |y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]|
    \]
Properties of LTI Systems by the Convolution

- Stability
  - To show that summability of $s$ is a necessary it must be shown that if $s = \infty$, then a bounded input can be found that will cause an unbounded output.
  
  \[
  x[n] = \begin{cases} 
  \frac{h^*[-n]}{|h[-n]|}, & h[n] \neq 0 \\
  0, & h[n] = 0
  \end{cases}
  \]

  - $x[n]$ is bounded by unity:
  - The output at $n=0$:
    \[
    y[0] = \sum_{k=-\infty}^{\infty} x[-k]h[k] = \sum_{k=-\infty}^{\infty} \frac{|h[k]|^2}{|h[k]|} = s
    \]
  - Therefore, if $s=\infty$, it is possible for a bounded input sequence to produce an unbounded output sequences.
Properties of LTI Systems Reflected in the Impulse Response

- The impulse responses of the systems can be computed by the response to $\delta[n]$
  - Idle delay $y[n] = x[n - n_d]$, $-\infty < n < \infty$, $h[n] = \delta[n - n_d]$
  - Moving average
    
    $$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n - k]$$
    $$h[n] = \begin{cases} 
    \frac{1}{M_1 + M_2 + 1}, & -M_1 \leq n \leq M_2 \\
    0, & \text{otherwise}
    \end{cases}$$
  - Accumulator
    
    $$y[n] = \sum_{k=-\infty}^{n} x[k]$$
    $$h[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 
    1, & n \geq 0 \\
    0, & n < 0
    \end{cases}$$
  - Forward difference $y[n] = x[n+1] - x[n]$
    • $h[n] = \delta[n+1] - \delta[n]$
  - Backward difference $y[n] = x[n] - x[n - 1]$
    • $h[n] = \delta[n] - \delta[n - 1]$
Properties of LTI Systems by the Convolution

• Causality

– In a causal systems the output $y[n_0]$ depends only on the input samples $x[n]$, for $n < n_0$. This definition implies the condition:

$$h[n] = 0 \quad \text{for} \quad n < 0$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

$$y[n] = \sum_{m=\infty}^{-\infty} x[n - m] h[m] = \sum_{m=-\infty}^{\infty} h[m] x[n - m] = h[n] \ast x[n]$$
Conversion of noncausal to a causal system by delay

- Forward difference system cascaded with delay system is noncausal
  - The same result is obtained if the sequence is first delayed and then compute forward difference
    \[ h[n] = (\delta[n+1] - \delta[n]) \ast \delta[n-1] \]
    \[ = \delta[n-1] \ast (\delta[n+1] - \delta[n]) \]
    \[ = \delta[n] - \delta[n-1] \]
  - The result is the same as to the impulse response of the backward difference
  - A noncausal FIR system can be made causal by cascading it with a sufficiently long delay.
Inverse system

- An accumulator is cascaded with a backward system

\[ h[n] = u[n] \ast (\delta[n] - \delta[n - 1]) \]
\[ = u[n] - u[n - 1] \]
\[ = \delta[n] \]

- The overall impulse response is the impulse

\[ h[n] \ast h_i[n] = h_i[n] \ast h[n] = \delta[n] \]

- The output is equal to input
- The backward difference system is the inverse of accumulator
- Inverse systems are useful when it is necessary to compensate the effects of a linear system.
Linear Constant-Coefficient Difference Equations (LCCDE)

- The input $x[n]$ and the output $y[n]$ satisfy an Nth-order linear constant-coefficient difference equation of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- Difference Equation Representation of the Accumulator
  - The output for $n-1$:
    $$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$
  - separating the term $x[n]$ from the sum
    $$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$
  - Substitution of equations
    $$y[n] = x[n] + y[n-1]$$
    $$y[n] - y[n-1] = x[n]$$

![Diagram of Accumulator](image)
Difference Equation Representation of the Moving-Average System

- the impulse response of the moving-average system with $M_1 = 0$:

$$h[n] = \frac{1}{(m_2 + 1)}(u[n] - u[n - M_2 - 1])$$

- We also have:

$$y[n] = \frac{1}{(M_2 + 1)} \sum_{k=0}^{M_2} x[n - k]$$

- The impulse response can also be represented as:

$$h[n] = \frac{1}{(M_2 + 1)}(\delta[n] - \delta[n - M_2 - 1]) * u[n]$$
Difference Equation Representation of the Moving-Average System

- Difference equation for the block diagram of the moving-average system:

\[ x_1[n] = \frac{1}{(M_2 + 1)} (x[n] - x[n - M_2 - 1]) \]

- The output of accumulator system is:

\[ y[n] - y[n - 1] = x_1[n] \]

- The impulse response can also be represented as:

\[ y[n] - y[n - 1] = \frac{1}{(M_2 + 1)} (x[n] - x[n - M_2 - 1]) \]
Recursive Computation of Difference Equation

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \]

• If the input \( x[n] \) and a set of auxiliary values, for \( n < 0 \) is specified, then \( y[n] \) for \( n \geq 0 \) can be determined recursively by (separate \( k=0 \)):

\[
y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]
\]

• \( y[n] \) for \( n < -N \) can be calculated by rearranging the main equation (assuming that the auxiliary conditions of \( y[-1], y[-2], \ldots, y[-N] \) are given) (separate \( k=N \))

\[
y[n-N] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_N} x[n-k]
\]
Recursive Computation of Difference Equation

- Example
  \[ y[n] = ay[n-1] + x[n] \]
    - \( x[n] = K\delta[n] \)
    - \( y[-1] = c \)
- For \( n > -1 \):
  \[ y[0] = ac + K \]
  \[ y[1] = ay[0] + 0 = a(ac + K) = a^2c + aK \]
  \[ y[2] = ay[1] + 0 = a(a^2c + aK) = a^3c + a^2k \]
  \[ y[3] = ay[2] + 0 = a(a^3c + a^2K) = a^4c + a^3k \]
  ...
  \[ y[n] = a^{n+1}c + a^nK \quad \text{for } n \geq 0 \]
Recursive Computation of Difference Equation

• For $n < 0$

\[ y[n-1] = a^{-1}(y[n] - x[n]) \]

or

\[ y[n] = a^{-1}(y[n+1] - x[n+1]) \]

\[ y[-2] = a^{-1}(y[-1] - x[-1]) = a^{-1}c \]

\[ y[-3] = a^{-1}(y[-2] - x[-2]) = a^{-1}a^{-1}c = a^{-2}c \]

\[ y[-4] = a^{-1}(y[-3] - x[-3]) = a^{-1}a^{-2}c = a^{-3}c \]

\[ y[n] = a^{n+1}c \quad \text{for } n \leq 1 \]

\[ y[n] = a^{n+1}c + Ka^n u[n] \quad \text{for all } n \]
Linear Constant Coefficient Difference Equation Solution

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \]

- Direct solution consists of two parts
  \[ y[n] = y_p[n] + y_h[n] \]
  - \( y_p[n] \) is the particular solution
  - \( y_h[n] \) is homogenous solution and it satisfies:
    \[ \sum_{k=0}^{N} a_k y[n-k] = 0 \]
    - \( y_h[n] \) is the solution of the equation with \( x[n] = 0 \)
- It can also be solved by z-transform
Linear Constant Coefficient Difference Equation Solution

• Homogenous solution
  – The sequence $y_h[n]$ is a member of a family of solutions of the form:
    \[ y_h[n] = \sum_{m=1}^{N} A_m \lambda_m^n \]
  – Substituting the above sequence into \( \sum_{k=0}^{N} a_k y[n-k] = 0 \) we have:
    \[ \sum_{k=0}^{N} a_k \lambda^{n-k} = 0 \]
  – Therefore, \( \lambda_m \) are roots of the above equation:
  – It is assumed that all \( N \) roots of the above polynomial Equation are distinct.
  – Since \( y_h[n] \) has \( N \) undetermined coefficients of \( a_k \), a set of \( N \) auxiliary conditions is required for the unique specification of \( y[n] \) for a given input of \( x[n] \)
Linear Constant Coefficient Difference Equation Solution

- Particular solution
  - \( y_p[n] \) is in the scaled form of the input:
    \[
    x[n] = A(\text{cons \ tan} t) \Rightarrow y_p[n] = k
    \]
    \[
    x[n] = A\lambda^n \Rightarrow y_p[n] = \beta\lambda^n
    \]
    \[
    x[n] = An^m \Rightarrow y_p[n] = k_0n^M + k_1n^{M-1} + \ldots + k_M
    \]
  - Substituting \( y_p[n] \) and \( x[n] \) into
    \[
    \sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]
    \]
  - Finding coefficients
Linear Constant Coefficient Difference Equation Solution

• Example: \( y[n] + y[n-1] - 6y[n-2] = x[n] \)
  
  – Input: \( x[n] = 8u[n] \)
  
  – Initial conditions: \( x[-1] = 2, \quad x[-2] = 1 \)

• Homogenous part:

\[
y[n] + y[n-1] - 6y[n-2] = 0, \quad y[n] = \lambda^n
\]

\[
\lambda^n + \lambda^{n-1} - 6\lambda^{n-2} = 0
\]

\[
\lambda^{n-2}(\lambda^2 + \lambda - 6) = 0
\]

\[
(\lambda + 3)(\lambda - 2) = 0
\]

\[
\lambda_1 = -3, \quad \lambda_2 = 2
\]

\[
y_h[n] = c_1(-3)^n + c_2(2)^n
\]
Linear Constant Coefficient Difference Equation Solution

- Particular part: \( y[n] + y[n-1] - 6y[n-2] = x[n] \)
  - Input, constant: \( x[n] = 8u[n] \)
  - \( y_p[n] = k \) \( x[-1] = 1, \quad x[-2] = -1 \)

\[
y[n] + y[n-1] - 6y[n-2] = x[n] \\
k + k - 6k = 8u[n] \\
-4k = 8 \\
k = -2
\]

\( y[n] = y_h[n] + y_p[n] \)

\[
= c_1(-3)^n + c_2(2)^n - 2
\]
Linear Constant Coefficient Difference Equation Solution

- \( y[n] + y[n-1] - 6y[n-2] = x[n] \), \( y[-1] = 1 \), \( y[-2] = -1 \)
- \( c_1(-3)^n + c_2(2)^n - 2 = 8 \)
- \( y[n] = c_1(-3)^n + c_2(2)^n - 2 \)
- \( n = 0 \), \( y[0] + y[-1] - 6y[-2] = x[0] \)
- \( c_1 + c_2 - 2 + 1 + 6 = 8 \)
- \( c_1 + c_2 = 3 \)

- \( n = 1 \), \( y[1] + y[0] - 6y[-1] = x[1] \)
- \( -3c_1 + 2c_2 - 2 + c_1 + c_2 - 2 - 6 = 8 \)
- \( -2c_1 + 3c_2 = 18 \)

- \( c_1 = -1.8 \), \( c_2 = 4.8 \)
- \( y[n] = -1.8(-3)^n + 4.8(2)^n - 2 \), \( n \geq 0 \)
Frequency-Domain Representation Of Discrete-time Signals And Systems

• Discrete-time signals may be represented in different ways

• Complex exponential sequences are eigenfunctions of linear time-invariant systems

• the response to a sinusoidal input is sinusoidal with the same frequency as the input and with amplitude and phase determined by the system.

• The representations of signals in terms of sinusoids or complex exponentials (i.e., Fourier representations) are very useful in linear system theory.
Eigenfunctions for Linear Time-Invariant Systems

- Consider an LTI system with the impulse response $h[n]$ and an input sequence:

$$x[n] = e^{j\omega n}, \quad -\infty < n < \infty$$

- The corresponding output is:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = e^{j\omega n} \left( \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right)$$

- If we define

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \quad y[n] = H\left(e^{j\omega}\right) e^{j\omega n}$$

- Consequently, $e^{j\omega n}$ which is the same as the input is appeared in the output. It is called eigenfunction of the system, and the associated eigenvalue is: $H\left(e^{j\omega}\right)$

- The eigenvalue is called the frequency response of the system and it describes the change in complex amplitude of a complex exponential input signal as a function of the frequency $\omega$. It consists of the real and imaginary parts

$$H\left(e^{j\omega}\right) = H_{\text{Re}}\left(e^{j\omega}\right) + jH_{\text{Im}}\left(e^{j\omega}\right) \quad H\left(e^{j\omega}\right) = |H\left(e^{j\omega}\right)| e^{j\angle H\left(e^{j\omega}\right)}$$
Frequency Response of the Ideal Delay System

• Consider the ideal delay system: \( y[n] = x[n - n_d] \)

• Consider input of: \( x[n] = e^{j\omega n} \)

• The output is: \( y[n] = e^{j\omega(n-n_d)} = e^{j\omega n} e^{-j\omega n_d} \)

• The frequency response: \( H(e^{j\omega}) = e^{-j\omega n_d} \)

• Other method by impulse response \( h[n] = \delta[n - n_d] \)

\[
H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d} = \cos(\omega n_d) + j \sin(\omega n_d)
\]

• The magnitude and phase are

\[
|H(e^{j\omega})| = 1, \quad \angle H(e^{j\omega}) = \omega n_d
\]
Sinusoidal Response of LTI Systems

- Consider a sinusoidal input:
  \[ x[n] = A \cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n} \]

  \[ x_1[n] = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} \quad y_1[n] = H(e^{j\omega_0}) \frac{A}{2} e^{j\phi} e^{j\omega_0 n} \]

  \[ x_2[n] = \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n} \quad y_2[n] = H(e^{-j\omega_0}) \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n} \]

  \[ y[n] = \frac{A}{2} [H(e^{j\omega_0}) e^{j\phi} e^{j\omega_0 n} + H(e^{-j\omega_0}) e^{-j\phi} e^{-j\omega_0 n}] \]

- If \( h[n] \) is real:
  \[ H(e^{-j\omega_0}) = H^*(e^{j\omega_0}) \]

- Therefore
  \[ y[n] = A \left| H(e^{j\omega_0}) \right| \cos(\omega_0 n + \phi + \theta) \quad \theta = \angle H(e^{j\omega_0}) \]

- Example: Ideal delay system
  \[ |H(e^{j\omega})| = 1, \quad \angle H(e^{j\omega}) = \omega n_d \]

  \[ y[n] = A \cos(\omega_0 n + \phi - \omega_0 n_d) \]

  \[ = A \cos[\omega_0 (n - n_d) + \phi] \]
Frequency Response of LTI systems

• The concept of the frequency response of LTI systems is essentially the same for continuous-time and discrete-time systems.

• An important difference is that the frequency response of discrete-time LTI systems is periodic.

\[ H(e^{j(\omega + 2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega + 2\pi)n} \]

\[ e^{-j(\omega + 2\pi)n} = e^{-j\omega n} e^{-j2\pi n} = e^{-j\omega n} \]

\[ H(e^{j(\omega + 2\pi)}) = H(e^{j\omega}) \]

\[ H(e^{j(\omega + 2\pi r)}) = H(e^{j\omega}) \quad r \text{ is an integer} \]
Representation of Sequences by Fourier Transforms

- Many sequences can be represented by a Fourier integral of the form:

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n} \]

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]

- \( x[n] \) is represented as a superposition of infinitesimally small complex sinusoids of the form:

\[ \frac{1}{2\pi} X(e^{j\omega})e^{j\omega n} \, d\omega \]

- \( X(e^{j\omega}) \) shows how much of each frequency component is required to synthesize \( x[n] \)
Impulse Response and Fourier Transform

- The frequency response of an LTI system is the Fourier transform of the impulse response.

\[ H(e^{j\omega}) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \]

- The impulse response can be obtained from the inverse Fourier transform of frequency response

\[ h[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H(e^{j\omega})e^{j\omega n} d\omega \]
Fourier Transform and Stability

- To represent a sequence \( x[n] \) by its IFT, \( X(e^{j\omega}) \) must be bounded or in the reverse respect the infinite sum must converge.
  \[
  \left| X(e^{j\omega}) \right| < \infty \quad \text{for all } \omega
  \]
- \( X(e^{j\omega}) \) is the limit as \( M \to \infty \) in the finite sum:
  \[
  X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}
  \]
- A sufficient condition for convergence can be found as follows:
  \[
  \left| X(e^{j\omega}) \right| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| |e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty
  \]
- If \( x[n] \) is absolutely summable then \( X(e^{j\omega}) \) exist and therefore all stable sequences have Fourier transforms.
- FIR systems are stable and therefore they have a finite, continuous frequency response. When a sequence has infinite length, we must be concerned about convergence of the infinite sum.
Absolute Summability for a Suddenly-Applied Exponential

• Consider the following sequence and its Fourier transform:

\[ x[n] = a^n u[n] \]
\[ X(e^{j \omega}) = \sum_{n=0}^{\infty} a^n e^{-j \omega n} = \sum_{n=0}^{\infty} (ae^{-j \omega})^n \]
\[ = \frac{1}{1 - ae^{-j \omega}} \quad \text{if } |a| < 1 \]

• Clearly, the condition \(|a| < 1\) is the condition for the absolute summability of \(x[n]\):

\[ \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty \quad \text{if } |a| < 1 \]

• Absolute summability is a sufficient condition for the existence of a Fourier transform representation, and it also guarantees uniform convergence.
Impulse Response of the Ideal Lowpass Filter

- The frequency response of a lowpass filter

\[ H_{lp} (e^{j\omega}) = \begin{cases} 
1 & |\omega| < \omega_c \\
0 & \omega_c < |\omega| \leq \pi 
\end{cases} \]

- The impulse response \( h_{lp}[n] \) can be found using the Fourier transform synthesis equation:

\[
h_{lp}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi jn} \left[ e^{j\omega n} \right]_{\omega_c}^{\omega_c} = \frac{1}{2\pi jn} (e^{j\omega n} - e^{-j\omega n})
\]

\[
= \frac{\sin \omega_c n}{\pi n} \quad -\infty < n < \infty
\]

- since \( h_{lp}[n] \) is nonzero for \( n < 0 \), the ideal lowpass filter is noncausal.
- Also, \( h_{lp}[n] \) is not absolutely summable.
- The sequence values approach zero as \( n \to \infty \), but only as \( 1/n \). This is because \( H_{lp} (e^{j\omega}) \) is discontinuous at \( \omega = \omega_c \) .
Symmetry Properties of the Fourier Transform

- A conjugate symmetric sequence:
  \[ x_e[n] = x_e^*[-n] \]

- Conjugate anti-symmetric sequence:
  \[ x_o[n] = -x_o^*[-n] \]

- Any sequence \( x[n] \) can be expressed as a sum of conjugate symmetric and conjugate anti-symmetric sequence:
  \[ x[n] = x_e[n] + x_o[n] \]
  \[ x_e[n] = \frac{1}{2} (x[n] + x^*[−n]) = x_e^*[−n] \]
  \[ x_o[n] = \frac{1}{2} (x[n] - x^*[−n]) = -x_o^*[−n] \]

- A real sequence that is conjugate symmetric is called an even sequence \( x_e[n] = x_e[−n] \)
- A real sequence that is conjugate antisymmetric is called an odd sequence.
  \[ x_o[n] = -x_o[−n] \]
Symmetry Properties of the Fourier Transform

• A Fourier transform $X(e^{j\omega})$ can be decomposed into a sum of conjugate-symmetric and conjugate antisymmetric functions as

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$$

$$X_e(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})]$$

$$X_o(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})]$$

• By substituting $-\omega$ for $\omega$ in above equations, it follows that $X_e(e^{j\omega})$ is conjugate symmetric and $X_o(e^{j\omega})$ is conjugate antisymmetric:

$$X_e(e^{j\omega}) = X_e^*(e^{-j\omega})$$

$$X_o(e^{j\omega}) = -X_o^*(e^{-j\omega})$$
## Symmetry Properties of the Fourier Transform

1. \(x^*[n]\)  
   \[X^*(e^{-j\omega})\]
2. \(x^*[-n]\)  
   \[X^*(e^{j\omega})\]
3. \(\Re\{x[n]\}\)  
   \[X_e(e^{j\omega})\] (conjugate-symmetric part of \(X(e^{j\omega})\))
4. \(j\Im\{x[n]\}\)  
   \[X_o(e^{j\omega})\] (conjugate-antisymmetric part of \(X(e^{j\omega})\))
5. \(x_e[n]\) (conjugate-symmetric part of \(x[n]\))  
   \[X_R(e^{j\omega}) = \Re\{X(e^{j\omega})\}\]
6. \(x_o[n]\) (conjugate-antisymmetric part of \(x[n]\))  
   \[jX_I(e^{j\omega}) = j\Im\{X(e^{j\omega})\}\]

**The following properties apply only when \(x[n]\) is real:**

7. Any real \(x[n]\)  
   \[X(e^{j\omega}) = X^*(e^{-j\omega})\] (Fourier transform is conjugate symmetric)
8. Any real \(x[n]\)  
   \[X_R(e^{j\omega}) = X_R(e^{-j\omega})\] (real part is even)
9. Any real \(x[n]\)  
   \[X_I(e^{j\omega}) = -X_I(e^{-j\omega})\] (imaginary part is odd)
10. Any real \(x[n]\)  
    \[|X(e^{j\omega})| = |X(e^{-j\omega})|\] (magnitude is even)
11. Any real \(x[n]\)  
    \[\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})\] (phase is odd)
12. \(x_e[n]\) (even part of \(x[n]\))  
    \[X_R(e^{j\omega})\]
13. \(x_o[n]\) (odd part of \(x[n]\))  
    \[jX_I(e^{j\omega})\]
Fourier Transform Theorems

1. \( ax[n] + by[n] \)  
   \( aX(e^{j\omega}) + bY(e^{j\omega}) \)

2. \( x[n - n_d] \)  \((n_d \text{ an integer})\)  
   \( e^{-j\omega n_d} X(e^{j\omega}) \)

3. \( e^{j\omega_0 n} x[n] \)  
   \( X(e^{j(\omega - \omega_0)}) \)

4. \( x[-n] \)  
   \( X(e^{-j\omega}) \)

5. \( nx[n] \)  
   \( X(e^{j\omega}) \)  
   \( X^*[e^{j\omega}] \) if \( x[n] \) real.
   \( \left. j \frac{dX(e^{j\omega})}{d\omega} \right| \)

6. \( x[n] \ast y[n] \)  
   \( X(e^{j\omega})Y(e^{j\omega}) \)

7. \( x[n]y[n] \)  
   \( \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta \)

Parseval's theorem:

8. \( \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \)

9. \( \sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega \)