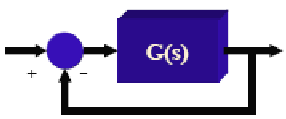
The Root Locus Method of Linear Feedback Systems

Ref: Chapter 7: **Dorf, R. C.** & **Bishop, R. H.** ,
Modern Control Systems
Chapter 8: **Nise, N. S.**
Control System Engineering



Introduction

➢ ***Root Locus illustrates how the poles of the closed-loop system vary with the closed-loop gain.***

➢ ***Graphically, the locus is the set of paths in the complex plane traced by the closed-loop poles as the root locus gain is varied from <u>zero to infinity.</u>***

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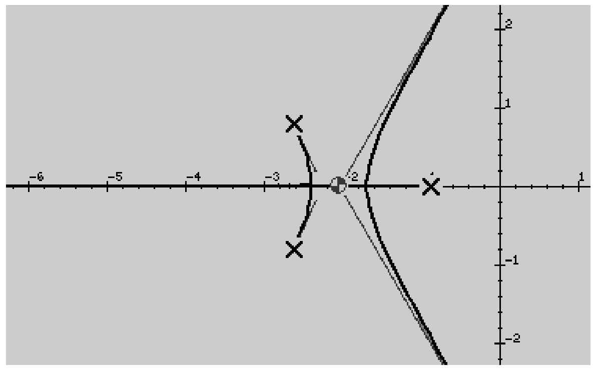
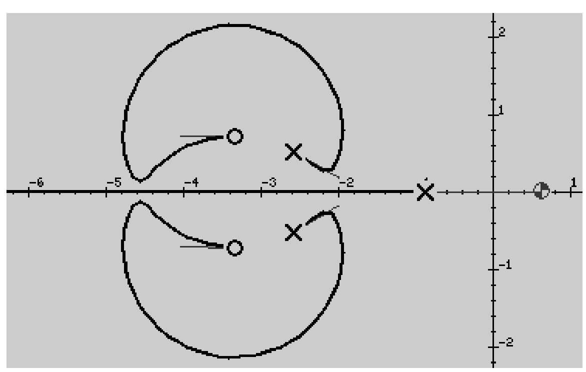
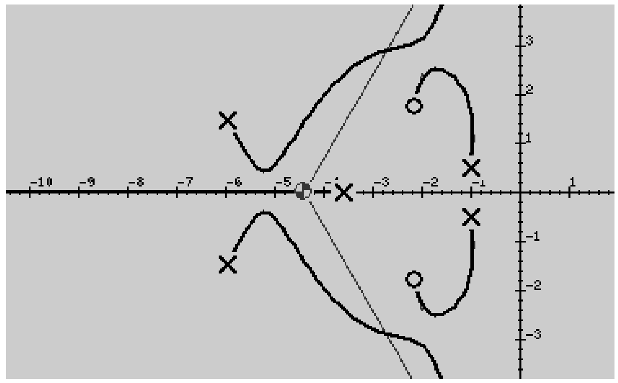
Example of Root Locus



Locus for a system with three poles and no zeros



Locus for a system with three poles and two zeros. Note that the part of the locus off the real axis is close to joining with the real axis, in which case break points would occur.



Locus for a system with 5 poles and 2 zeros.

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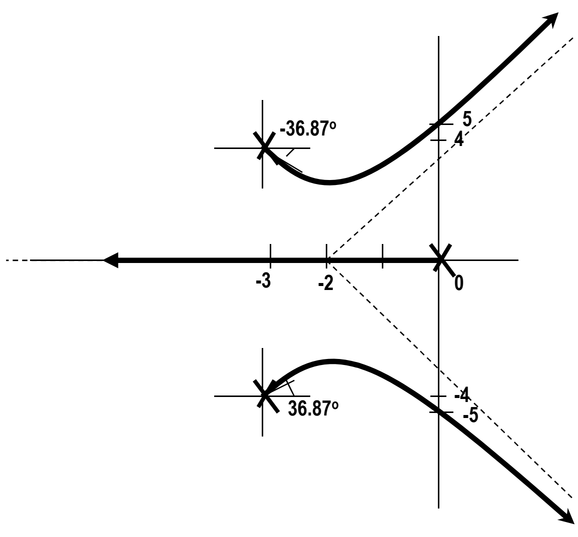

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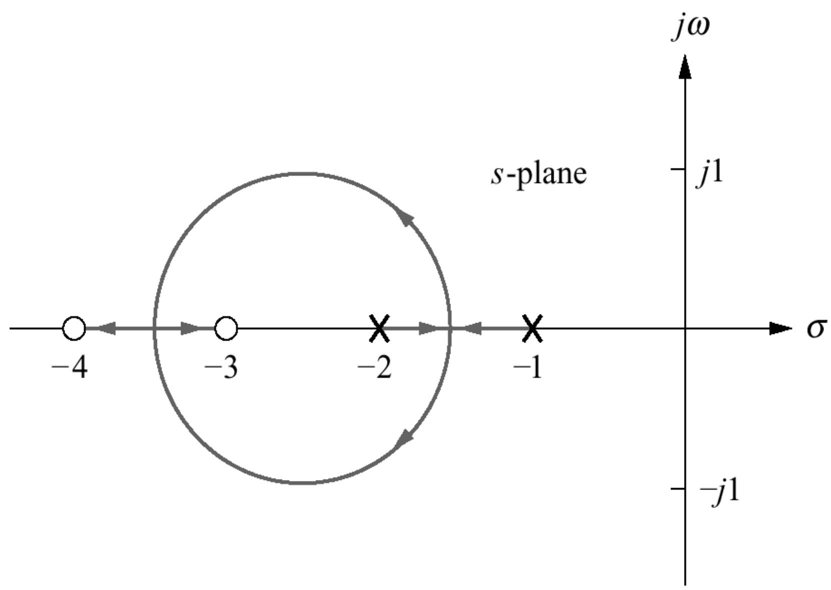

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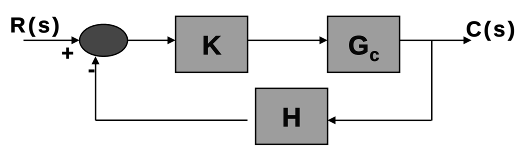

block diagram of the closed loop system

given a forward-loop transfer function $KG_c(s)H(s)$

where *K* is the root locus gain, and the corresponding closed-loop transfer function

$$G(s)=\frac{KG_c(s)H(s)}{1+KG_c(s)H(s)}$$

the root locus is the set of paths traced by the roots of

$$1 + KG_c(s)H(s) = 0$$

➢ ***as K varies from zero to infinity. As K changes, the solution to this equation changes.***

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So basically, the root locus is sketch based on the *characteristic equation* of a given transfer function.

Let say:

$$G(s)=\frac{KG_c(s)H(s)}{1+KG_c(s)H(s)}$$

Thus, the characteristic equation :

$$C.E = 1+KG_c(s)H(s)$$

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Root locus starts from the characteristic equation.

$$1+KG_c(s)H(s)=0$$

Split into 2 equations:

$$KG_c(s)H(s)=-1+j0$$

Magnitude condition

$$|KG_c(s)H(s)|=|1|$$

Angle condition

$$\angle KG_c(s)H(s)=180^o(1+2m)$$

where $m = 0,1,2,\cdots$

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Remarks

If S_i is a root of the characteristic equation, then 1+G(s)H(s) = 0 *OR* Both the magnitude and Angle conditions must be satisfied

If not satisfied→ Not part of root locus

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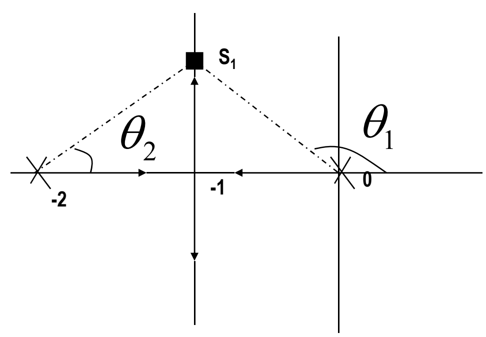
Let say my first search point, S_1

$$S_1 = -1 + j1$$



(A) Satisfy the angle condition FIRST

$$\angle G(s)|_{s=-1+j1} = -\theta_1 - \theta_2 = -135^o - 45^o = -180^o$$

(B) Magnitude condition to find K

$$\left| \frac{K}{(-1+j1)(-1+j1+2)} \right| = 1$$

$$K = (\sqrt{2})(\sqrt{2}) = 2$$

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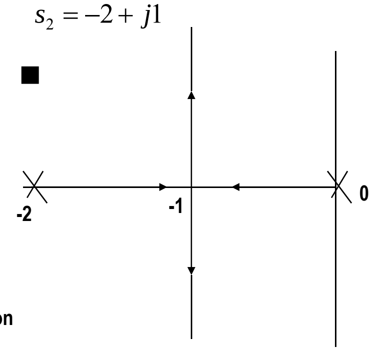
Let say the next search point $s_2 = -2 + j1$



Check whether satisfy angle condition

$$G(s)|_{s=-2+j1} = \frac{K}{(-2+j1)(-2+j1+2)} = \frac{K}{(-2+j1)(j1)}$$

$$\angle G(s)|_{s=-2+j1} = -[180^o - \tan^{-1}(1/2) - 90^o = -270^o + \tan^{-1}(1/2)$$

$$\angle G(s)|_{s=-2+j1} \neq -180^o \, OR \, +180^o$$

Since angle condition was not satisfied → not part of root locus

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CONSTRUCTION RULES OF ROOT LOCUS

8 RULES TO FOLLOW

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Construction Rules of Root Locus

$$KG(s)H(s) = -1$$

$$KG(s)H(s) = \frac{K \prod_{i=1}^{M} (s + z_i)}{\prod_{j=1}^{N} (s + p_j)}$$

Then

$$KG(s)H(s) = \frac{K \prod_{i=1}^{M} (s + z_i)}{\prod_{j=1}^{N} (s + p_j)} = -1$$

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Thus

$$K = -\frac{\prod (s + p_j)}{\prod (s + z_i)}$$

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Rule 1 : When K = 0

$$s = -p_j$$

→ Root at open loop poles

(a) The R-L starts from open loop poles
(b) The number of segments is equal to the number of open loop poles

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Rule 2: When K = ∞

$$s = -z_i$$

(a) The R-L terminates (end) at the open loop zeros

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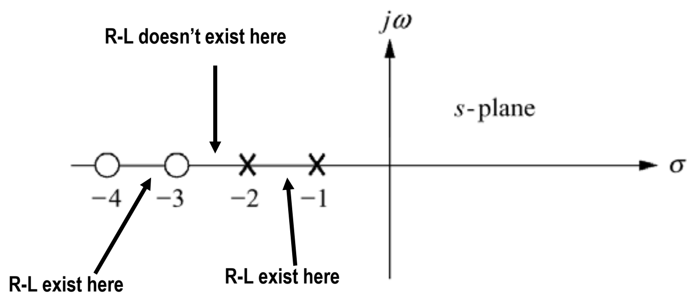
Rule 3: Real-axis segments

On the real axis, for K > 0 the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros

Example 2

$$G(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)}$$



Rule 4: Angle of asymptote

$$\phi_A = \frac{180^o(1+2m)}{N_P - N_Z}, m = 0,1,\cdots(N_P - N_Z - 1)$$

N_P = number of poles

N_Z = number of zeros

Example 3

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

$N_P=3$

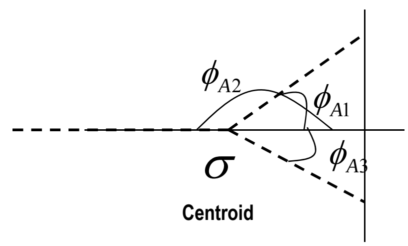
$N_Z=0$

$m = 0,1,2$

$$\phi_{A1} = \frac{180^o(1+2(0))}{3-0} = 60^o$$

$$\phi_{A2} = \frac{180^o(1+2(1))}{3-0} = 180^o$$

$$\phi_{A3} = \frac{180^o(1+2(2))}{3-0} = 300^o = -60^o$$



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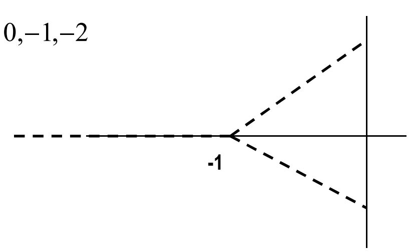
Rule 5: Centroid

$$\sigma_A = \frac{\sum \mathrm{Re}(p_j) - \sum \mathrm{Re}(z_i)}{N_P - N_Z}$$

From example 3 $\quad G(s) = \frac{K}{s(s+1)(s+2)}$

Recall $\quad s = \mathrm{Re} + j\,\mathrm{Im}$ $\quad$ Thus, $\quad s = 0,-1,-2$

$$\sigma_A = \frac{-1-2}{3} = -1$$



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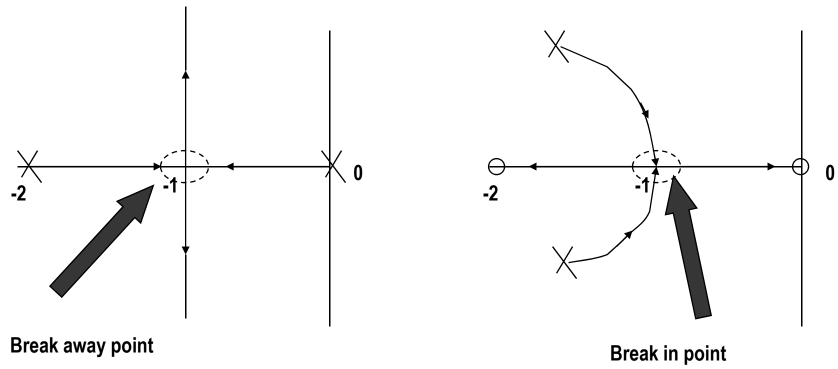
Rule 6: Break away & break in points (if exist)



How to find these points ?

$$KG(s)H(s) = -1$$

$$K = \frac{-1}{G(s)H(s)}$$

Differentiate K with respect to S & equate to zero

$$\frac{dK}{ds} = \frac{d}{ds}\left[\frac{-1}{G(s)H(s)}\right] = \frac{d}{ds}\left[G(s)H(s)\right] = 0$$

How ?

Solve for S → this value will either be the break away or break in point

Let say

$$G(s)H(s) = \frac{N_1 N_2}{D_1 D_2}$$

$$\frac{d}{ds}\left[G(s)H(s)\right] = \frac{D_1 D_2 (N_1 N_2)' - N_1 N_2 (D_1 D_2)'}{(D_1 D_2)^2} = 0$$

Thus,

$$D_1 D_2 (N_1 N_2)' - N_1 N_2 (D_1 D_2)' = 0$$

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Example 4

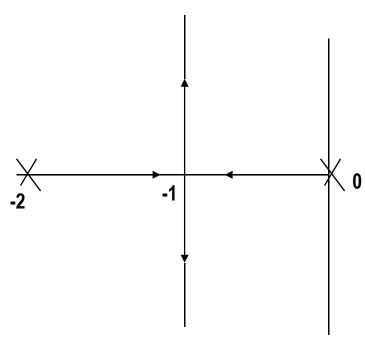
$$G(s) = \frac{1}{s(s+2)}$$

Solution:

$$\frac{dK}{ds} = \frac{d}{ds}\left[G(s)\right] = \frac{d}{ds}\left[\frac{1}{s(s+2)}\right] = 0$$

$$= \frac{-(2s+2)}{(s(s+2))^2} = 0$$

$$s = -1$$



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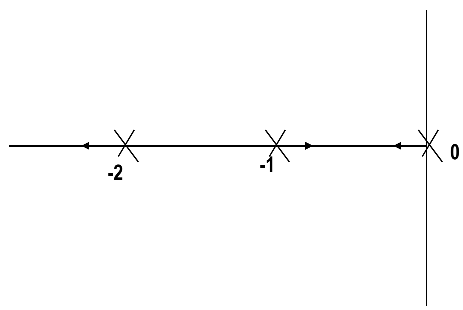
Example Root Locus

$$G(s) = \frac{1}{s(s+1)(s+2)}$$



$N_P = 3$

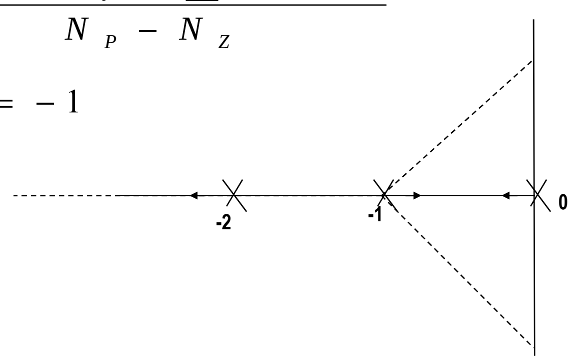
$N_Z = 0$

$$\phi_A = \frac{180^o(1+2m)}{N_P - N_Z}, m = 0,1,\cdots(N_P - N_Z - 1)$$

$$\phi_A = 60^o, 180^o, -60^o$$

$$\sigma_A = \frac{\sum \mathrm{Re}(p_j) - \sum \mathrm{Re}(z_i)}{N_P - N_Z}$$

$$= \frac{-1-2}{3} = -1$$



$$\frac{dK}{ds} = \frac{d}{ds}\left[\frac{1}{s^3 + 2s^2 + 2s}\right]$$

$$= \frac{-1(3s^2 + 6s + 2)}{(s^3 + 3s^2 + 2s)^2} = 0$$

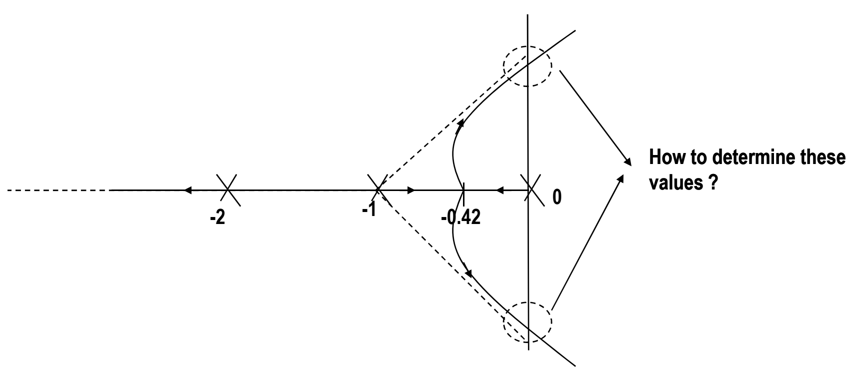
$$(3s^2 + 6s + 2) = 0$$

$$s_1, s_2 = \frac{-6 \pm \sqrt{36 - 4(3)(2)}}{2(3)}$$

Break away point

$$s_1, s_2 = -1.6, -0.42$$

Invalid, why ???



Rule 7 : R-L crosses jω-axis (if exist)

If there is a breakaway/ break in point

Use Routh Hurwitz

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From characteristic equation

$$1 + KG(s) = 0$$

$$1 + K\left[\frac{1}{s(s+1)(s+2)}\right] = 0$$

$$s^3 + 3s^2 + 2s + K = 0$$

Construct the Routh array

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S^3	1	2
S^2	3	K
S^1	6-K	
S^0	K	

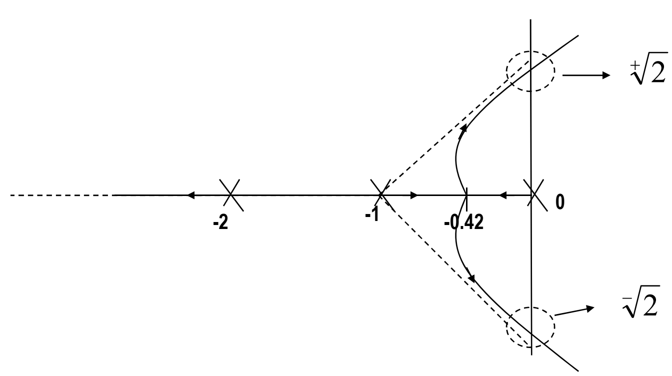
Force S^1 row to zero or K=6

Replace K=6 into S^2 row

$$3s^2 + 6 = 0$$
$$s^2 = -2$$
$$s = j\pm\sqrt{2}$$

Since

$$s = j\omega$$
$$j\omega = j\pm\sqrt{2}$$
$$\omega = \pm\sqrt{2}$$



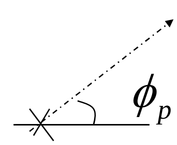
Rule 8 : Angle of departure (arrival) (if exist)

No complex conjugate poles/zeros in this example

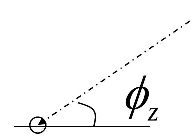
Rule 8 : Angle of departure (arrival) (if exist)

Finding angles of departure and arrival from complex poles & zero

Root locus start from open loop poles → **Angle of departure**



Root locus terminates at open loop zeros → **Angle of arrival**



Angle of Departure

$$\phi_p = 180^o + \sum \angle z_i - \sum \angle p_j$$

Angle of Arrival

$$\phi_z = 180^o + \sum \angle p_j - \sum \angle z_i$$

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For sketching Root Locus

8 rules to follow:

Rule 1 : When K = 0

Rule 2 : When K = ∞

Rule 3 : Real axis segment

Rule 4: Angle of asymptote

Rule 5 : Centroid

Rule 6 : Break away & break in point

Rule 7 : R-L crosses the jω-axis

Rule 8 : Angle of Departure / Arrival

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Example 1

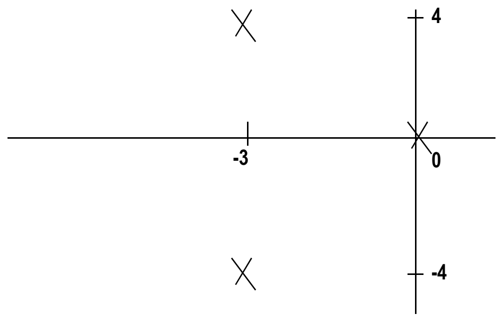
$$G(s) = \frac{K}{s(s^2 + 6s + 25)}$$

Step 1: Determine poles & zeros

poles at : $0, \frac{-6 \pm \sqrt{36 - 100}}{2} = 0, -3 \pm j4$

$N_p=3$
$N_z=0$

Step 2: Maps poles & zeros on the s-plane



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Step 3: Calculate angle of asymptote

$$\phi_A = \frac{180^o(1+2m)}{N_P - N_Z}, m = 0,1,\cdots(3-0-1) = 0,1,2$$

$$\phi_{A1} = \frac{180^o(1+2(0))}{3-0} = 60^o$$

$$\phi_{A2} = \frac{180^o(1+2(1))}{3-0} = 180^o$$

$$\phi_{A3} = \frac{180^o(1+2(2))}{3-0} = 300^o = -60^o$$

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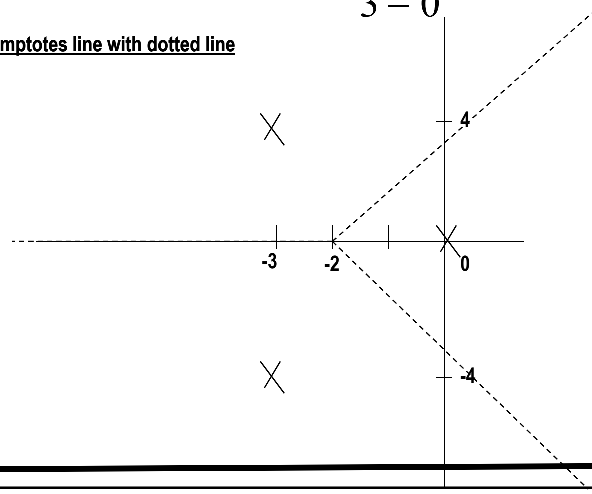
Step 4: Determine centroid

$$\sigma_A = \frac{\sum \mathrm{Re}(p_j) - \sum \mathrm{Re}(z_i)}{N_P - N_Z}$$

poles: $= 0, -3 \pm j4$

$$\sigma_A = \frac{-3-3}{3-0} = -2$$

Step 5: Sketch asymptotes line with dotted line



Step 6: Determine the break-away or break in points

$$\frac{dK}{ds} = \frac{d}{ds}\left[-\frac{1}{G(s)}\right] = \frac{d}{ds}\left[-s(s^2+6s+25)\right] = 0$$

$$= -[3s^2 + 12s + 25] = 0$$

$$s_1, s_2 = \frac{-12 \pm \sqrt{144-300}}{6} = \frac{-12 \pm \sqrt{-156}}{6}$$

Complex numbers

NO break away or break in points
→ only exist on real-axis of s-plane

Step 7: Determine jω-crossing

$$G(s) = \frac{K}{s(s^2 + 6s + 25)}$$

From Characteristic equation:

$$1 + G(s) = 0$$

$$1 + \frac{K}{s(s^2 + 6s + 25)} = 0$$

$$s^3 + 6s^2 + 25s + K = 0$$

Construct the Routh array

s^3	1	25
s^2	6	K
s^1	(6(25)-K)/6= (150-K)/6	
s^0	K	

Force row s^1 to be zero, thus 150-K=0; K = 150

Substitute K = 150 into row s^2

$$6s^2 + 150 = 0$$

$$s^2 = -25$$

$$s = \pm\sqrt{-25} = \pm j5$$

Since s = jω

$$j\omega = \pm j5$$

Thus $$\omega = \pm 5$$

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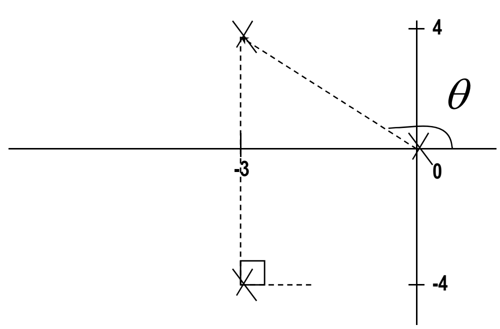
Step 8: Since complex poles exist, calculate the angle of departure

$$\phi_p \,|_{s=-3+j4} = 180^o + \sum \angle z_i - \sum \angle p_j$$



$$\phi_p = 180^o \underbrace{-\theta}_{\text{due to } s=0} \underbrace{-90^o}_{\text{due to its complex conjugate}}$$

due to s=0

due to its complex conjugate

How to determine θ?→

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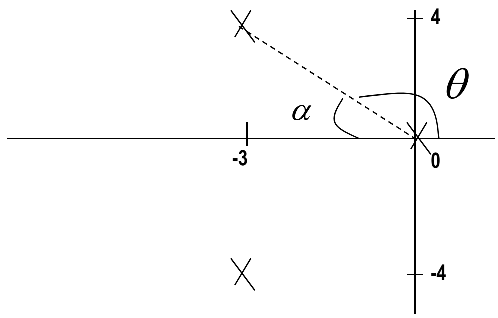
To determine θ

$$\theta = 180^o - \alpha$$



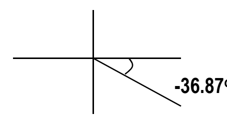
$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

Thus:

$$\theta = 180^o - \tan^{-1}\left(\frac{4}{3}\right) = 126.87^o$$

<u>Angle of departure :</u>

$$\phi_p = 180^o - 126.87 - 90^o = -36.87^o$$



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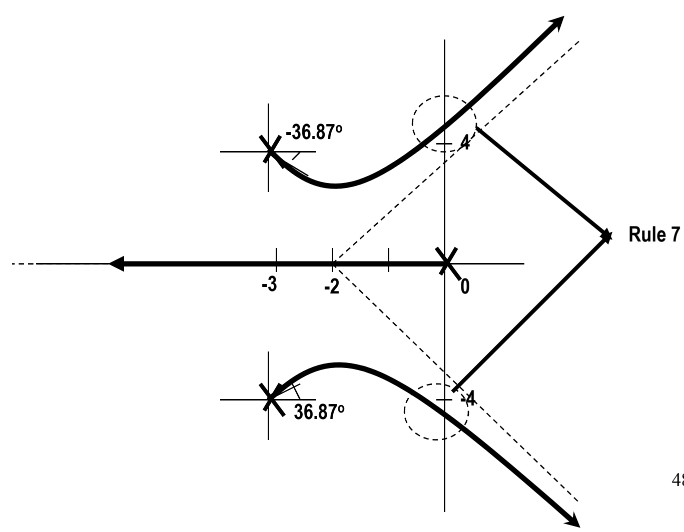
<u>**Step 9: Sketch the root locus**</u>



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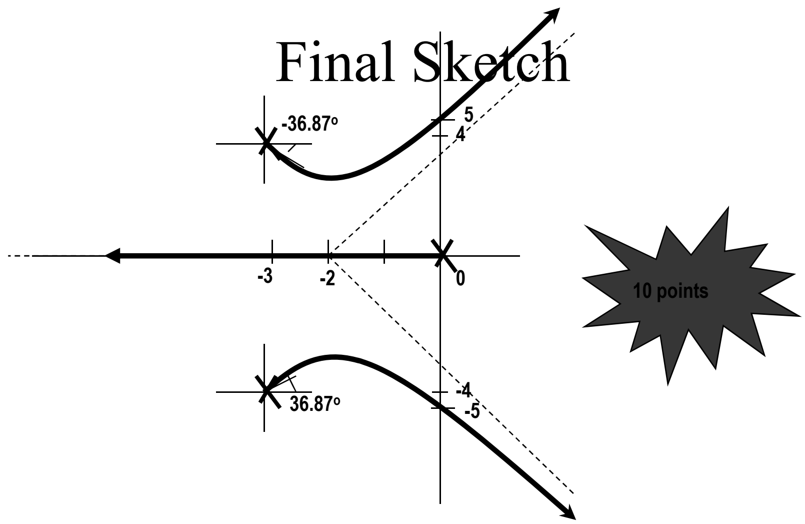
Final Sketch



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Try it

Sketch the root locus for:

$$G(s) = \frac{K(s^2 + 2s + 2)}{s(s+1)(s+2)}$$

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How to use Matlab to sketch root locus

$$KG(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)} = \frac{K[s^2 + 7s + 12]}{s^2 + 3s + 2}$$

In Matlab:

```
num = [1 7 12]
den = [1 3 2]
rlocus(num,den)
```

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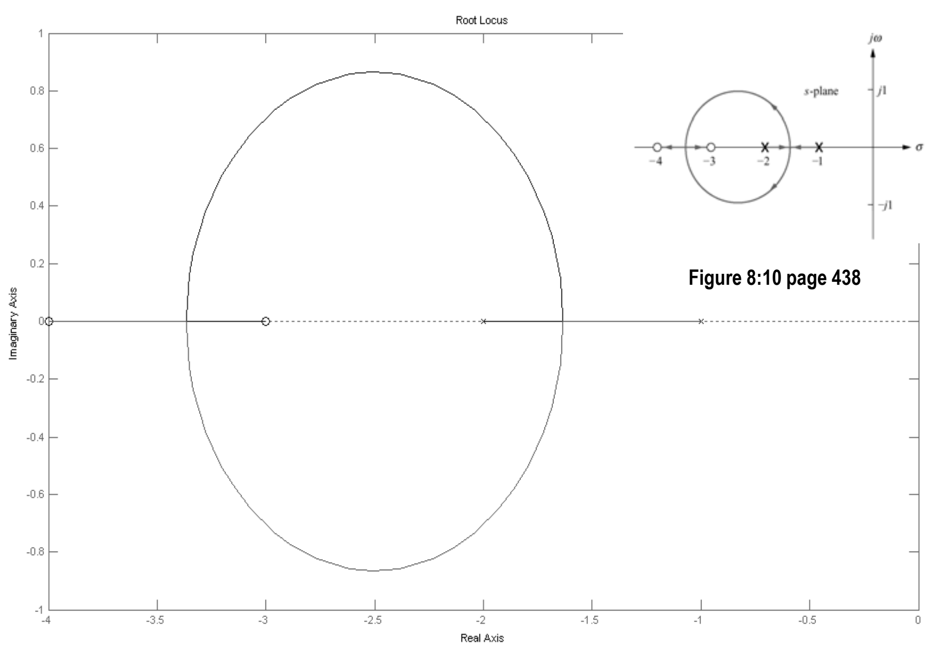

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